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CHAPTER 9

Inequalities and linear programming

- What is a linear inequality?
- How do we solve linear inequalities?
- What is linear programming and how is it used?

In Chapter 3, 'Linear graphs and models', you learned how linear equations and their graphs are used to model practical situations, such as plant growth, service charges and flow problems. In this chapter you will learn how linear inequalities and their graphs can be used to model a different set of practical situations, such as determining the mix of products in a supermarket to maximise profit, or designing a diet to provide maximum nutrition for minimum cost. This is known as **linear programming**. Linear programming requires you to solve both linear equations and linear inequalities. You learned how to solve linear equations in Chapter 2, 'Linear relations and equations'. You now need to learn how to solve linear inequalities.

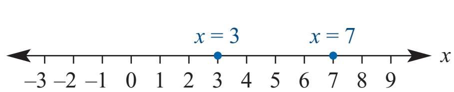
9.1 Linear inequalities in one variable

Linear inequalities in one variable and the number line

An expression such as $9 \leq 3x \leq 21$ is called a **linear inequality** in one variable. It is an *inequality*, not an equation, because it involves an inequality sign ($\leq$) rather than an equals sign ($=$). The sign '$\leq$' means '*less than or equal to*'.

The solution to the linear *equation* $3x = 9$ is $x = 3$, and the solution to the linear *equation* $3x = 21$ is 7.

We can represent these solutions on a number line by putting a **closed circle** (•) on the number line at $x = 3$ and $x = 7$ as shown.



When solving an *inequality* and graphing its solution on a number line, we need to be careful about whether the end values of the solution are included in the range of possible values.

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End values included

To solve the linear inequality

$$9 \le 3x \le 21$$

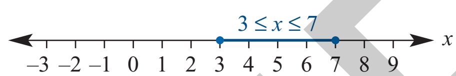
we divide through by 3 and get

$$\frac{9}{3} \le \frac{3x}{3} \le \frac{21}{3}$$

or $$3 \le x \le 7$$

There is no single solution to this inequality. Any value of x from 3 to 7 is a solution. For example, $x = 3$, $x = 3.5$, $x = 4.95$ and $x = 7$ are all possible solutions. In fact, it is impossible to list every possible solution, as there are an infinite number of solutions. However, we can represent all the *possible* solutions on a number line.

This is done by marking the points $x = 3$ and $x = 7$ with a closed circle (●) on the number line. These points are then joined by drawing a solid line to indicate that all the values between $x = 3$ and $x = 7$ are also solutions, as shown below.



End values not included

To solve the linear inequality

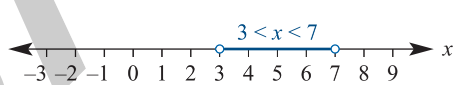
$$9 < 3x < 21$$

we divide through by 3 to obtain the solution

$$3 < x < 7$$

The sign '$<$' means '*less than*'. This means that $x = 3$ and $x = 7$ are *not* solutions, but all values between $x = 3$ and $x = 7$ are possible solutions.

To represent this solution on a number line, mark in the points $x = 3$ and $x = 7$ with an **open circle** (○). These two open circles are then joined by a solid line to indicate that all the values *between* 3 and 7 are solutions, but *not* $x = 3$ and $x = 7$.



Note that $7 > x > 3$ represents the same values of x as $3 < x < 7$.

A gallery of signs

$=$ as in $a = b$	reads as	'a equals b'
$>$ as in $a > b$	reads as	'a is greater than b'
$\ge$ as in $a \ge b$	reads as	'a is greater than or equal to b'
$<$ as in $a < b$	reads as	'a is less than b'
$\le$ as in $a \le b$	reads as	'a is less than or equal to b'

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Example 1 Solving an inequality and graphing the solution

Solve the inequality

$$-10 < 5x \leq 40$$

for x and display the solution on a number line.

Solution

1 Write the inequality.

$$-10 < 5x \leq 40$$

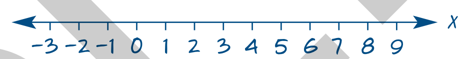
2 Solve the inequality for x by dividing through by 5.

$$\text{or } \frac{-10}{5} < \frac{5x}{5} \leq \frac{40}{5}$$

3 Display the solution on a number line.

$$\text{or } -2 < x \leq 8$$

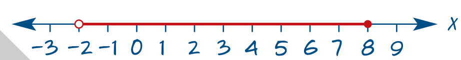
- Draw a number line to include −2 and 8.



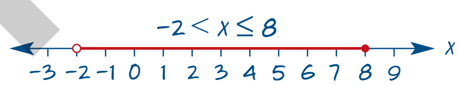
- Mark the point $x = -2$ with an open circle.

- Mark the point $x = 8$ with a closed circle.

- Join the two points with a solid line.



- Write in the solution inequality on the graph.



Example 2 Solving an inequality and graphing the solution

Solve the inequality

$$10x > 20$$

for x and display the solution on a number line.

Solution

1 Write the inequality.

$$10x > 20$$

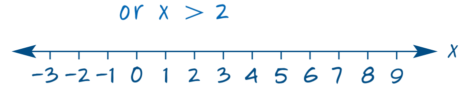
2 Solve the inequality for x by dividing through by 10.

$$\text{or } \frac{10x}{10} > \frac{20}{10}$$

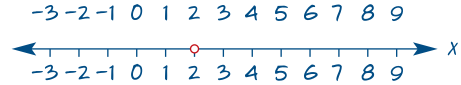
3 Display the solution on a number line.

$$\text{or } x > 2$$

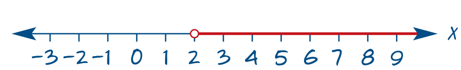
- Draw a number line to include 2.



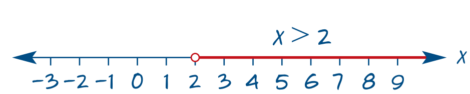
- Mark in the point $x = 2$ with an open circle.



- To indicate all values of x greater than 2, draw a solid line from this point to the right that potentially goes on forever.



- Write in the solution inequality on the graph.



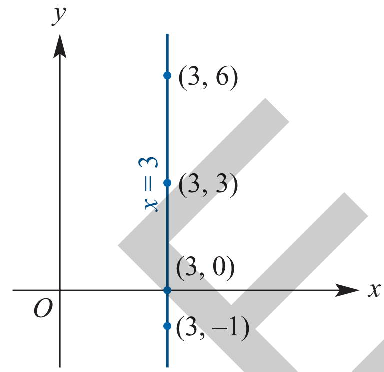
Linear inequalities in one variable and the coordinate plane

We can also represent linear inequalities in one variable on the coordinate plane.

If the equation $x = 3$ is plotted on a set of axes we will have a vertical straight line, located at $x = 3$.

While the value of y changes along the line, for every point on this line the value of x is 3.

Just as with graphing the solution of an inequality on a number line, we need to be careful about whether the boundary lines (end values) of the solution are included in the range of possible values.

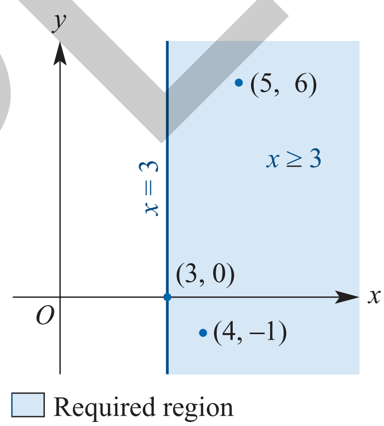


Boundary line included

If we tried to plot the inequality $x \geq 3$, we would have to plot all the points in the plane that have an x-value greater than or equal to 3.

Of course we cannot show each individual point. What we do is shade in the region containing these points. The shaded region starts at the vertical line $x = 3$ and extends right forever.

Some representative points that satisfy the condition $x \geq 3$, and which are found in the shaded region, have also been plotted.

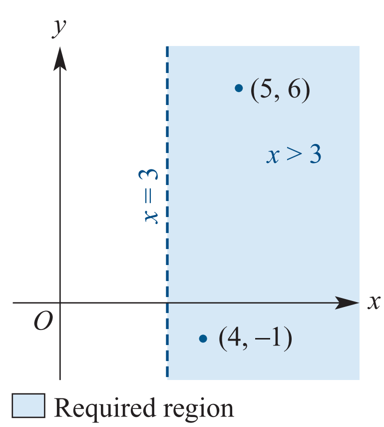


Boundary line not included

The plot of the inequality $x > 3$ is similar to the plot of $x \geq 3$, but the line $x = 3$ is drawn as a **dashed line** to indicate that it is *not* included in the region.

Some representative points that satisfy the condition $x > 3$ have also been plotted.

Note: For $(5, 6)$, $5 > 3$
$(4, -1)$, $4 > 3$
Therefore both points satisfy the inequality $x > 3$.



Example 3 Plotting a linear inequality in one variable on the coordinate plane

On the coordinate plane, plot the graphs of:

a $y \leq 4$

b $-1 < y < 3$

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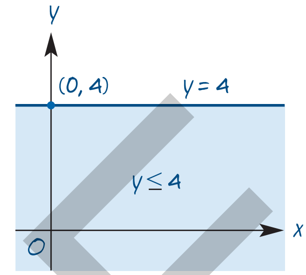
Solution

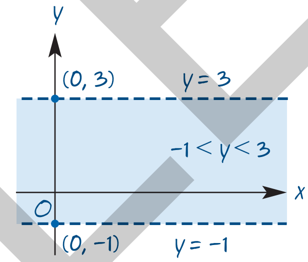
a $y \leq 4$

1. Draw in a solid line $y = 4$ to define the boundary of the shaded region.
2. Shade the region on and below the line $y = 4$ to represent all the points defined by $y \leq 4$.



b $-1 < y < 3$

1. Draw in a dashed line $y = 3$ to define the upper boundary of the shaded region.
2. Draw in a dashed line $y = -1$ to define the lower boundary of the shaded region.
3. Shade the region between the lines $y = 3$ and $y = -1$ to represent all the points defined by $-1 < y < 3$.



Exercise 9A

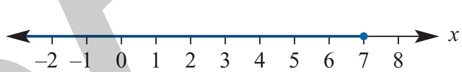
1 Which of the symbols $<$, $=$ or $>$ should be placed in the box in each of the following?

a $7 \square 9$ **b** $3 \square 2$ **c** $7 + 1 \square 9 - 1$ **d** $0.5 \square 1$
e $8 \square 4$ **f** $-3 \square 1$ **g** $-2 \square -1$ **h** $0 \square 0.5$

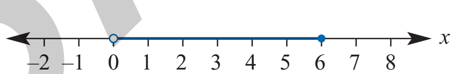
2 Represent each of the following inequalities on a number line.

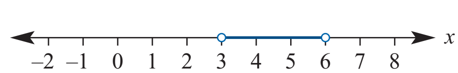
a $1 \leq x \leq 4$ **b** $0 < x < 4$ **c** $x < 4$
d $x \geq 4$ **e** $-1 \leq x < 4$ **f** $3 < x \leq 5$

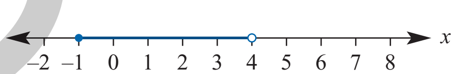
3 Write down an inequality represented by each of the following graphs.

a


b


c


d


e


4 Solve each of the following inequalities and represent its solution on a number line.

a $3x \geq 15$ **b** $20x < 100$ **c** $2x > -4$
d $9x \geq 36$ **e** $-12 \leq 6x < 24$ **f** $10 < 5x \leq 25$

5 A person becomes a teenager when they turn 13. They stop being a teenager when they turn 20. Let x be the variable age (in years).

a Write down an inequality in terms of x that defines a teenager.
b Graph this inequality on a number line.

6 Carry-on luggage in most passenger aircraft can weigh no more than 5 kg. Let x be the variable weight (in kg).

a Write down an inequality in terms of x that defines the acceptable weight for carry-on luggage.
b Graph this inequality on a number line.

7 Graph the following inequalities on the coordinate plane.

a $x \le 1$	**b** $x > -2$	**c** $y \le 5$
d $y > 1$	**e** $x < 2$	**f** $-2 \le y \le 2$
g $-1 < x < 2$	**h** $3 < x \le 5$	**i** $-3 \le y < 0$

9.2 Linear inequalities in two variables

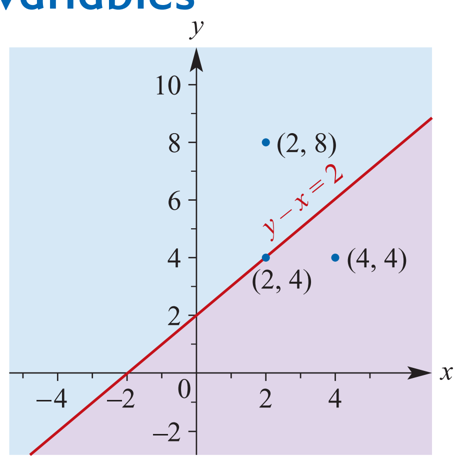
The inequalities

$$y - x > 2 \quad y - x \ge 2$$

$$y - x < 2 \quad y - x \le 2$$

are linear inequalities in two variables, x and y.

To help us interpret these inequalities, we have drawn the graph of $y - x = 2$. This line (red) separates the coordinate plane into two regions.



The line $y - x = 2$

The **line** is defined by the equation $y - x = 2$ and coloured red. It includes all the points that lie *on* the line.

From the graph above we can see that the point (2, 4) lies on the line. The points (2, 8) and (4, 4) clearly do *not* lie on the line.

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We can also show this by carrying out the following tests, using the equation of the line.

Test:

(2, 4): $y - x = 4 - 2 = 2$; so the point (2, 4) lies on the line $y - x = 2$.

(2, 8): $y - x = 8 - 2 = 6$; 6 is greater than 2, so (8, 9) does *not* lie on $y - x = 2$.

(4, 4): $y - x = 4 - 4 = 0$; 0 is less than 2, so (4, 4) does *not* lie on the line $y - x = 2$.

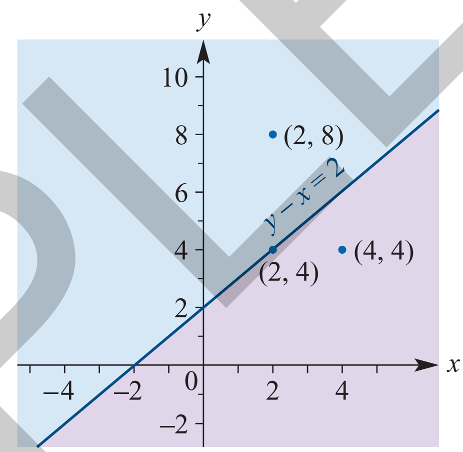
The regions $y - x > 2$ and $y - x \geq 2$

This region is defined by the inequality $y - x > 2$ and coloured light blue. It includes all the points that lie *above* the line; the point (2, 8) is an example.

By *including the line* in this region, we have a way of representing the inequality

$$y - x \geq 2$$

This region includes all the *points on and above* the line.



From the graph on the right we can see that

- the points (2, 4) and (2, 8) are examples of points that lie in this region.
- the point (4, 4) clearly does *not* lie in the region.

We can also show this by carrying out the following tests, using the equation of the line.

Test:

(2, 4): $y - x = 4 - 2 = 2$; so the point (2, 4) lies in the region $y - x \geq 2$.

(2, 8): $y - x = 8 - 2 = 6$; 6 is greater than 2, so the point (2, 8) lies in the region $y - x \geq 2$

(4, 4): $y - x = 4 - 4 = 0$; 0 is less than 2, so (4, 4) does *not* lie in the region $y - x \geq 2$.

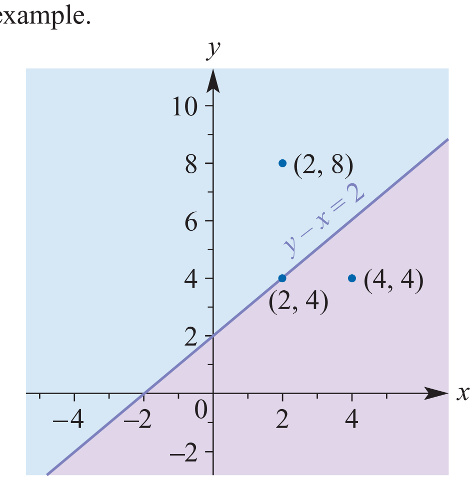
The regions $y - x < 2$ and $y - x \leq 2$

This region is defined by the inequality $y - x < 2$ and coloured purple. It includes all the points that lie *below* the line; the point (4, 4) is an example.

By *including the line* in this region, we have a way of representing the inequality

$$y - x \leq 2$$

This region includes *all the points on and below* the line.



From the graph on the right we can see that

- the points (2, 4) and (4, 4) are examples of points that lie in this region.
- the point (2, 8) clearly does *not* lie in the region.

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We can also show this by carrying out the following tests using the equation of the line.

Test:

(2, 4): $y - x = 4 - 2 = 2$; so the point (2, 4) lies in the region $y - x \leq 2$.

(2, 8): $y - x = 8 - 2 = 6$; 6 is greater than 2, so (8, 9) does *not* lie in the region $y - x \leq 2$.

(4, 4): $y - x = 4 - 4 = 0$; 0 is less than 2, so (4, 4) lies in the region $y - x \leq 2$.

We now have a graphical way of representing inequalities.

Linear inequalities can be represented by regions in the coordinate plane

If the inequality sign is:

- $\leq$ or $\geq$, the line defining the region is *included*, indicated by using a **solid line** to indicate the boundary
- $<$ or $>$, the line defining the region is *not included*, indicated by using a **dashed line** to indicate the boundary.

Graphing a linear inequality in two variables

Example 4 **Graphing a linear inequality in two variables**

Sketch the graph of the region $3x + 2y \leq 18$.

Solution

1 Find the intercepts for the boundary line $3x + 2y = 18$.

- Find the y-intercept. Substitute $x = 0$ into the equation and solve for y.
- Find the x-intercept. Substitute $y = 0$ into the equation and solve for x.

$3x + 2y = 18$

When $x = 0$, $2y = 18$

$y = 9$

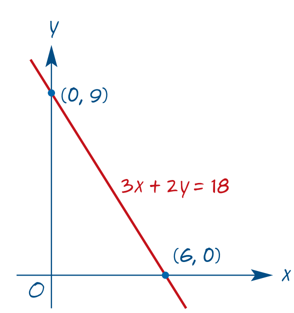
$\therefore$ y-intercept is (0, 9).

When $y = 0$, $3x = 18$

$x = 6$

$\therefore$ x-intercept is (6, 0).

2 On a labelled set of axes, draw a straight line through the two intercepts. Use a solid line to indicate that the line is included in the region. Label the line.



3 Use a test point to determine whether the required region lies above or below the line.

Note: The origin (0, 0) is usually a good point to test.

Test (0, 0):

$3x + 2y = 3(0) + 2(0) = 0$

$0 < 18$, so (0, 0) lies in the region $3x + 2y \leq 18$.

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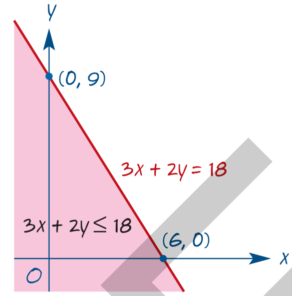
4 As (0, 0) is below the line, the required region lies on and below the line. Shade in the region on and below the line. Label the region.



Example 5 Graphing a linear inequality in two variables

Sketch the graph of the region $4x - 5y > 20$.

Solution

1 Find the intercepts for the boundary line $4x - 5y = 20$.

- Find the y-intercept. Substitute $x = 0$ into the equation and solve for y.
- Find the x-intercept. Substitute $y = 0$ into the equation and solve for x.

$4x - 5y = 20$

When $x = 0$, $-5y = 20$

$y = -4$

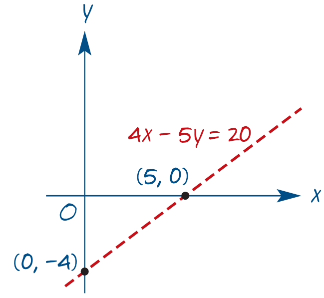
$\therefore$ y-intercept is $(0, -4)$.

When $y = 0$, $4x = 20$

$x = 5$

$\therefore$ x-intercept is $(5, 0)$.

2 On a labelled set of axes, draw a straight line through the two intercepts. Use a dashed line to indicate that the boundary line is *not* included in the region. Label the line.



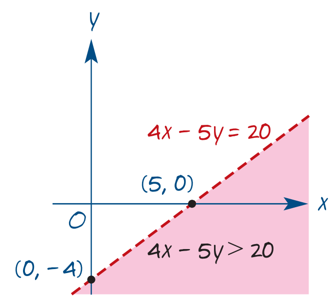
3 Use a test point to determine whether the required region lies above or below the line.

Test (0, 0):

$4x - 5y = 4(0) - 5(0) = 0$

$0 < 20$, so (0, 0) does not lie in the region $4x - 5y > 20$.

4 As (0, 0) is above the line, the required region lies below the line. Shade in the region on and below the line. Label the region.



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Summary: Plotted linear inequalities

- Graph the inequality as if it contained an equals (=) sign.
- Draw a solid line if the inequality is $\leq$ or $\geq$.
- Draw a dashed line if the inequality is $<$ or $>$.
- Pick a point not on the line to use as a test point. The origin is a good test point, provided the boundary line does not pass through the origin.
- Substitute the test point into the inequality. If the point makes the inequality true, shade the region containing the test point. If not, shade the region *not* containing the test point.

Exercise 9B

1 Test to see whether the point (0, 0) lies in the following regions.

a $x + y \geq 0$ **b** $x + y < 4$ **c** $2x + y > 2$
d $3x - 2y \geq 3$ **e** $y - 2x > 5$ **f** $x - 3y < 6$

2 Test to see whether the point (1, 2) lies in the following regions.

a $x + y \geq 0$ **b** $x + y < 0$ **c** $2x + y > 2$
d $3x - 2y \geq 3$ **e** $2x + 3y > 5$ **f** $5y - 2x \geq 8$

3 Graph the following inequalities.

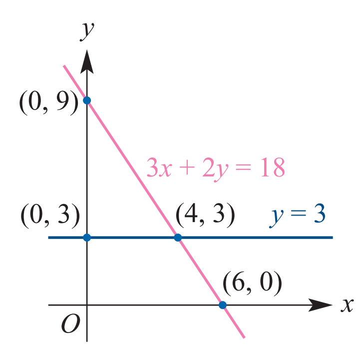
a $y - x \leq 5$ **b** $2x - y \leq 4$ **c** $x - y < 3$
d $x + y \geq 10$ **e** $3x + y \leq 9$ **f** $5x + 3y \geq 15$
g $3y - 5x < 15$ **h** $2y - 5x > 5$ **i** $y - x > -3$

9.3 Feasible regions

In Chapter 2, 'Linear relations and equations', you learned how to solve pairs of simultaneous linear equations graphically.

For example, to solve the pair of *linear equations*

$$3x + 2y = 18$$
$$y = 3$$



graphically, we simply plot their graphs and find the point of intersection.

The *solution* is the *point* on the coordinate plane that is *common to both graphs*. This is the point (4, 3), the point where the two lines intersect. From this, we concludes that $x = 4$ and $y = 3$.

When we try to solve the pair of simultaneous *linear inequalities*

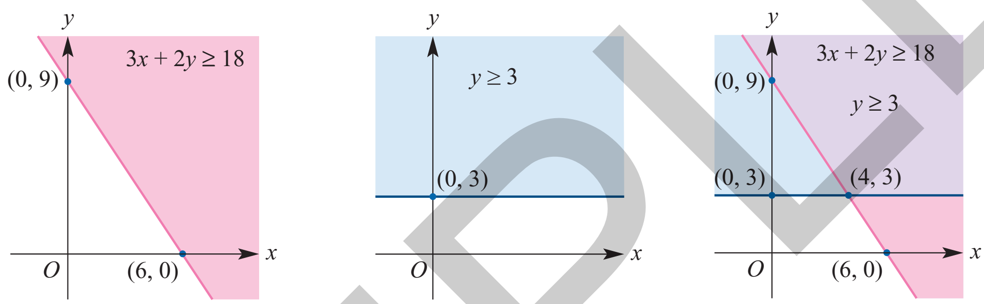
$$3x + 2y \geq 18$$
$$y \geq 3$$

graphically, there is not a single solution, but many solutions. The *solutions* are all the points that lie in the *region* in the coordinate plane that is *common to both inequalities*.

The region common to both inequalities is called the **feasible region**. It is called the feasible region because all the points in this region are possible solutions of the pair of simultaneous linear inequalities.

The feasible region (solution region) for a set of inequalities is determined by finding the region common to all of the inequalities involved. This process is illustrated below for the inequalities

$$3x + 2y \geq 18 \quad \text{and} \quad y \geq 3.$$



The region shaded pink is defined by the inequality $3x + 2y \leq 18$.

The region shaded blue is defined by the inequality $y \geq 3$.

The region shaded purple is the **feasible region**. It is the region common to the inequalities $3x + 2y \leq 18$ and $y \geq 3$.

The method we have used to graphically determine the feasible region is called **shading in**. Sometimes this method of finding the region common to a set of inequalities can quickly become messy and impractical when we have too many inequalities. Fortunately, for the sort of applications you will meet in this chapter, the required region will lie in the first quadrant and involve only a small number of inequalities so that the 'shading in' method is appropriate.

Example 6 Graphing a feasible region

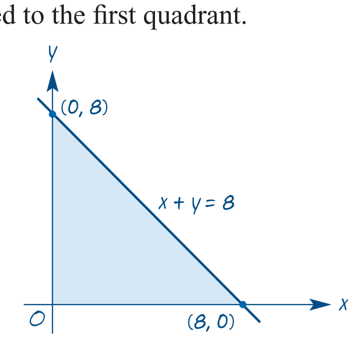
Graph the feasible region for the following four simultaneous inequalities:

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 8, \quad 3x + 5y \leq 30$$

Solution

Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant.

1 Graph the inequality $x + y \leq 8$ in the first quadrant.
- Plot the boundary line $x + y = 8$, marking and labelling the y-intercept (0, 8) and the x-intercept (8, 0).
- Shade in the region bounded by the x- and y-axes and the line. Here it has been shaded blue.



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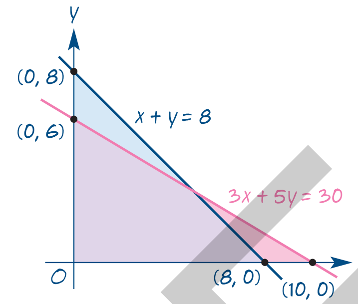
2 Graph the inequality $3x + 5y \le 30$ in the first quadrant.

- Plot the boundary line $3x + 5y = 30$, marking and labelling the y-intercept (0, 6) and the x-intercept (10, 0).
- Shade in the region bounded by the x- and y-axes and the line. Here it has been shaded pink, but it becomes purple where it overlaps the blue region.

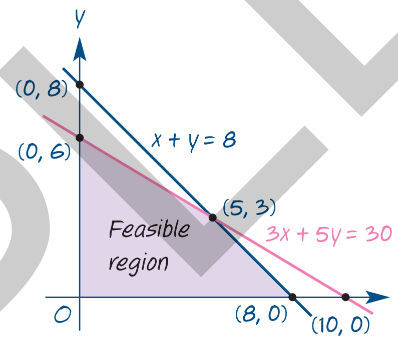


3 The overlap region (purple) is the feasible region.

- Label the overlap region the 'Feasible region'.
- To complete the feasible region, find the coordinates of the point where the two boundary lines intersect, by solving the simultaneous equations

$$x + y = 8$$
$$3x + 5y = 30$$

The lines intersect at the point (5, 3). Mark this point on the graph.



Example 7 Graphing a feasible region

Graph the feasible region for the following four simultaneous inequalities:

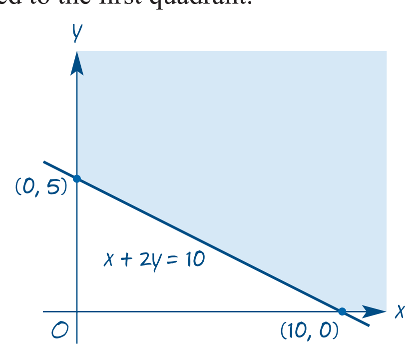
$$x \ge 0, \quad y \ge 0, \quad x + 2y \ge 10, \quad 6x + 4y \ge 36$$

Solution

Because $x \ge 0$ and $y \ge 0$, the feasible region is restricted to the first quadrant.

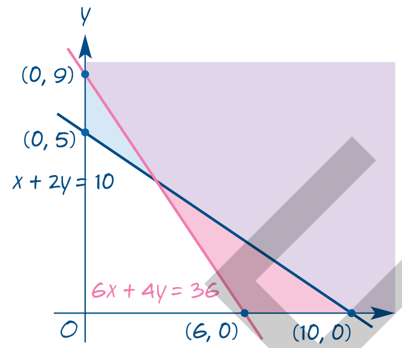
1 Graph the inequality $x + 2y \ge 10$ in the first quadrant.

- Plot the boundary line $x + 2y = 10$, marking and labelling the y-intercept (0, 5) and the x-intercept (10, 0).
- Shade in the region bounded by the x- and y-axes and the line. Here it has been shaded blue.



2 Graph the inequality $6x + 4y \leq 36$ in the first quadrant.

- Plot the boundary line $6x + 4y = 36$, marking and labelling the y-intercept (0, 9) and the x-intercept (6, 0).
- Shade in the region bounded by the x- and y-axes and the line. Here it has been shaded pink, but it becomes purple where it overlaps the blue region.

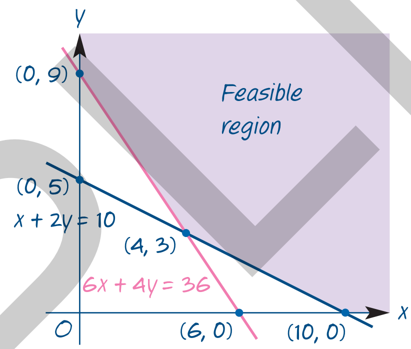


3 The overlap region (purple) is the feasible region.

- Label the overlap region the 'Feasible region'.
- To complete the feasible region, find the coordinates of the point where the two boundary lines intersect, by solving the simultaneous equations

$$x + 2y = 10$$
$$6x + 4y = 36$$

The lines intersect at the point (4, 3). Mark this point on the graph.



How to graph a feasible region using a TI-Nspire CAS

Graph the feasible region for the following four simultaneous inequalities:

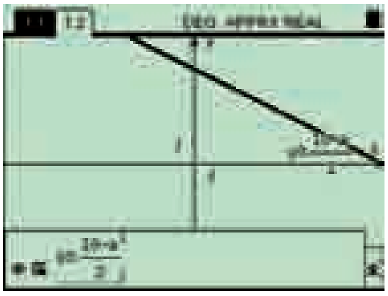
$$x \geq 0, \quad y \geq 0, \quad x + 2y \geq 10, \quad 6x + 4y \geq 36$$

Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant. We take this into account when setting the viewing window on the calculator.

Steps

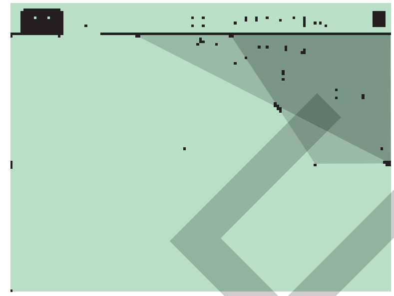
1 To graph the inequalities $x + 2y \geq 10$ and $6x + 4y \geq 36$ using a graphics calculator, first we need to rearrange both inequalities so that y is the subject. Hence,

$$x + 2y \geq 10 \text{ becomes } y \geq \frac{(10 - x)}{2}$$
$$6x + 4y \geq 36 \text{ becomes } y \geq \frac{(36 - 6x)}{4}$$



2 Open a new document (by pressing ctrl + N) and select **2: Graphs & Geometry**.

a Use the backspace key (clear) to delete the $f1(x) =$ and type in $y >= (10 - x) \div 2$. Press enter. This plots the inequality $x + 2y \geq 10$.

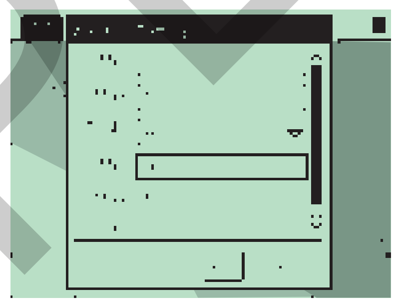


b Repeat the above but this time type in $y >= (36 - 6x) \div 4$. Press enter. This plots the inequality $6x + 4y \geq 36$

c Press ctrl + G to hide the entry line.

d The inequalities $x \geq 0$ and $y \geq 0$ indicate that the feasible region is restricted to the first quadrant. This is best achieved by resetting the viewing window.

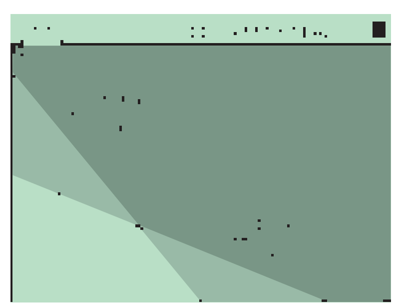
3 To reset the viewing window, press menu **/4:Window/1:Window Settings**. Using tab to move between the entry boxes, enter the following values:



- **XMin: 0**
- **XMax: 12**
- **XScale: Auto**
- **YMin: 0**
- **YMax: 10**
- **YScale: Auto**

4 Pressing enter confines the plot to the first quadrant. The graphs will appear as shown. The feasible region is the more heavily shaded region.

Note: It may be necessary to grab and move the graph labels if they overlap with other labels.



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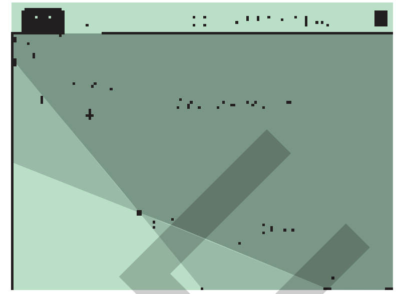
5 To complete the feasible region, we need to know the coordinates of the corner points.

a Press (menu) **/6:Points & Lines/3:Intersection Point/s.**

b Move the cursor to one of the graphs and press (click). Now move to the other graph and press (click). The point of intersection (4, 3) will be displayed. Press (esc) to exit the **Intersection Point** tool.

The other two points, (0, 9) and (10, 0), can be determined from the equations of the boundary lines.



How to graph a feasible region using the ClassPad

Graph the feasible region for the following four simultaneous inequalities:

$$x \geq 0, \quad y \geq 0, \quad x + 2y \geq 10, \quad 6x + 4y \geq 36$$

Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant. We take this into account when setting the viewing window on the calculator.

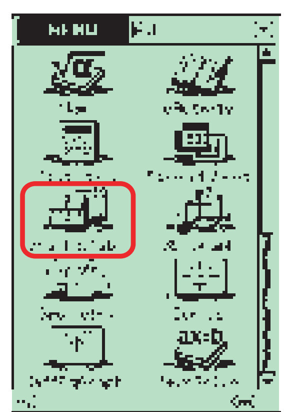
Steps

1 **From the application** menu, locate and open the **Graph and Table** (Graph&Tab...) **built-in application**.

To graph the inequalities $x + 2y \geq 10$ and $6x + 4y \geq 36$, **first we need to rearrange both inequalities so that *y* is the subject. Hence,**

$$x + 2y \geq 10 \text{ becomes } y \geq \frac{(10 - x)}{2}$$

$$6x + 4y \geq 36 \text{ becomes } y \geq \frac{(36 - 6x)}{4}$$



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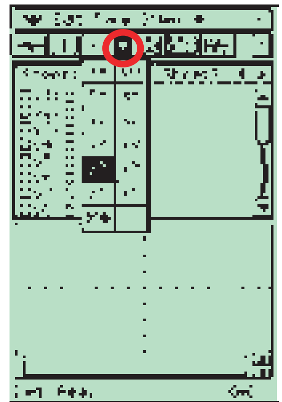
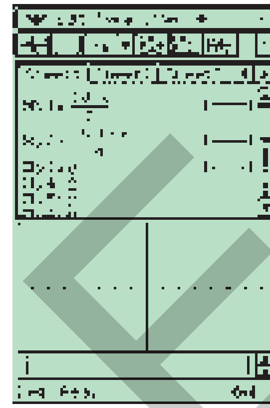
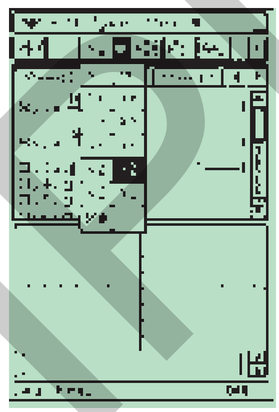
2 To set the calculator to draw the correct inequality, tap the down arrow (▾) adjacent to [y=] in the toolbar and select [y≧].
Adjacent to **y1:** type $(10 - x)/2$.
Press (EXE).
Adjacent to **y2:** type $(36 - 6x)/4$.
Press (EXE).
Adjacent to **y3:** type **0**.
Press (EXE).

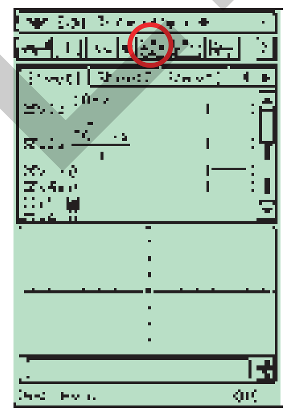


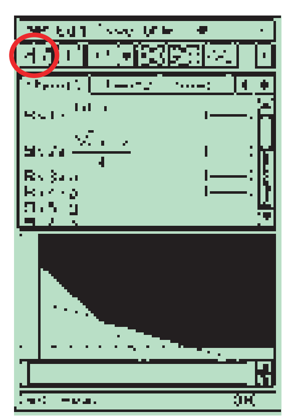


3 To enter the $x \geq 0$ inequality, tap the down arrow (▾) adjacent to [y≧] in the toolbar and select [x≧].
Adjacent to **y4:** type **0**.
Press (EXE).

Tap on the **View Window** icon ([⊞]) in the toolbar to set the graph viewing window.







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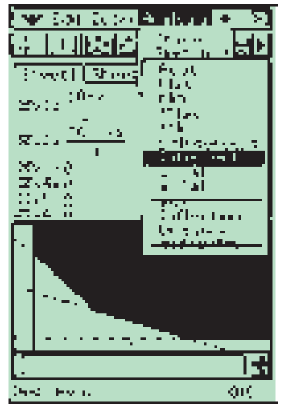
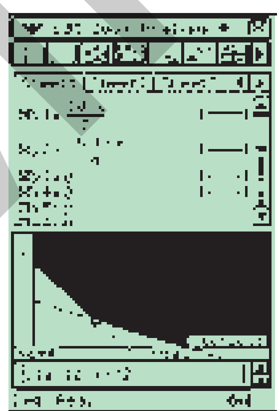
4 **To complete the region, the corner points need to be found.**
From the Analysis menu item, select G-solve, then Intersect.

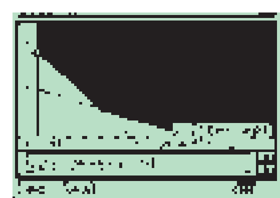


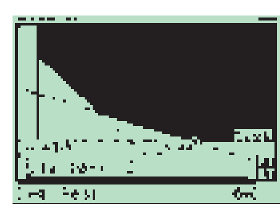
Use the up ▲ and ▼ directions from the blue oval directional button on the front of the calculator to select the equations for **y1** and **y2**.
When an equation has been selected, press (EXE) to confirm its choice.
The equation of each line is displayed in a window at the bottom of the graphing screen.

5 After the second equation has been selected and confirmed, the intersection point will be displayed on the screen, indicated by a cursor in the shape of a small cross. In this case, (4, 3).

The other two boundary points are (0, 9) and (10, 0).







Exercise 9C

Graph the feasible region for each of the following sets of linear inequalities.

1 $x \geq 0,\ y \geq 0,\ x + y \leq 10$

2 $x \geq 0,\ y \geq 0,\ 2x + 3y \leq 12$

3 $x \geq 0,\ y \geq 0,\ 3x + 5y \geq 15$

4 $x \geq 0,\ y \geq 0,\ x + y \leq 6,\ 2x + 3y \leq 15$

5 $x \geq 0,\ y \geq 0,\ 3x + y \leq 6,\ x + 2y \leq 7$

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6 $x \geq 0,\ y \geq 0,\ 5x + 2y \geq 20,\ 5x + 6y \geq 30$

7 $x \geq 0,\ y \geq 0,\ 4x + y \geq 12,\ 3x + 6y \geq 30$

8 $x \geq 0,\ y \geq 0,\ 2x - y \geq 0,\ x + y \leq 30$

9.4 Linear programming

Objective functions and constraints

An **objective function** is a quantity that you are trying to make as large as possible (for example, profits) or as small as possible (for example, the amount of material needed to make a dress). Of course, there are always factors, such as the resources available or the requirements of the dress pattern, that limit how much profit you can make or how little material you can use to make a dress. These are called **constraints**.

The linear programming problem

The process of **maximising** or **minimising** a linear quantity, subject to a set of constraints, is at the heart of linear programming.

The linear programming problem

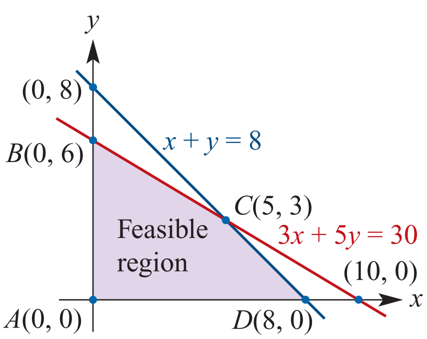
From the mathematical point of view, **linear programming** can be viewed as finding the point, or points, in a feasible region that gives the maximum or minimum value of some linear expression.

Finding the maximum value of an objective function

The aim is to find the maximum value of the objective function $P = 2x + 3y$, subject to the constraints:

$$x \geq 0,\ y \geq 0$$
$$x + y \leq 8$$
$$3x + 5y \leq 30$$

as shown by the feasible region opposite.



At first, this seems like an insurmountable problem, as there is an infinite number of points in the region to choose from. Fortunately, we can make use of the **corner point principle** to help us solve the problem.

The corner point principle

In linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region.

Note: If two corners give the same maximum or minimum value, then all points along a line joining the two points will also give the same maximum or minimum value.

This means that we only need to evaluate the objective function at each of the corner points, labelled A, B, C and D, and find which gives the maximum value. It helps to set up a table as follows.

Points	Objective function $P = 2x + 3y$
A (0, 0)	$P = 2 \times 0 + 3 \times 0 = 0$
B (0, 6)	$P = 2 \times 0 + 3 \times 6 = 18$
C (5, 3)	$P = 2 \times 5 + 3 \times 3 = 19$
D (8, 0)	$P = 2 \times 8 + 3 \times 0 = 16$

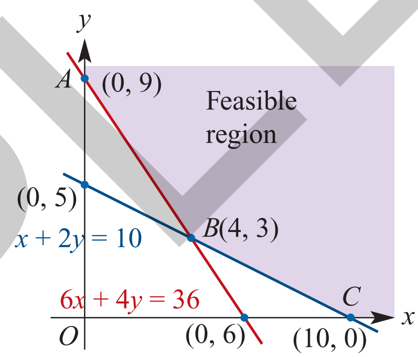
Thus, the maximum value of the objective function, $P = 19$, occurs when $x = 5$ and $y = 3$.

Example 8 Finding the minimum value of an objective function

Find the **minimum** value of the objective function $C = 5x + 2y$, subject to the constraints:

$$x \geq 0, y \geq 0$$
$$x + 2y \geq 10$$
$$6x + 4y \geq 36$$

as displayed in the feasible region opposite.



Solution

1 Set up a table for the objective function.

2 Evaluate the objective function at each of the corners A, B and C.

3 Identify the corner point giving the minimum value and write your answer.

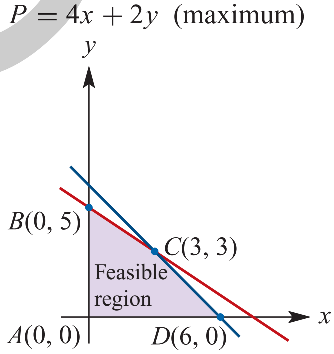
Points	Objective function $C = 5x + 2y$
A(0, 9)	$C = 5 \times 0 + 2 \times 9 = 18$
B(4, 3)	$C = 5 \times 4 + 2 \times 3 = 26$
C(10, 0)	$C = 5 \times 10 + 2 \times 0 = 50$

The minimum value is $C = 18$, which occurs when $x = 0$ and $y = 9$.

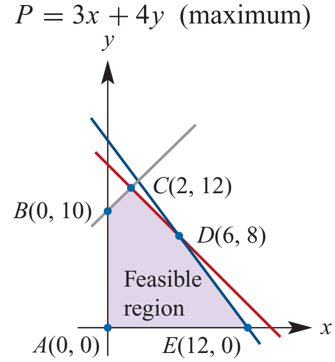
Exercise 9D

For each of the following objective functions and feasible regions, find the maximum or minimum value (as required) and the point at which it occurs.

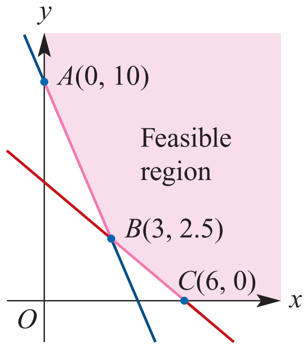
1 $P = 4x + 2y$ (maximum)



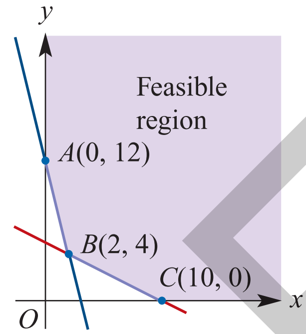
2 $P = 3x + 4y$ (maximum)



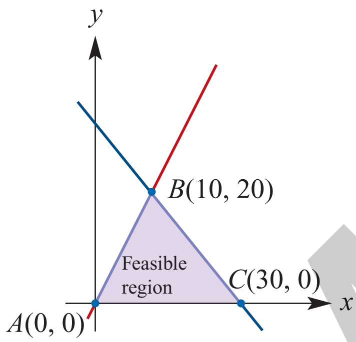
3 $C = 3x + 5y$ (minimum)



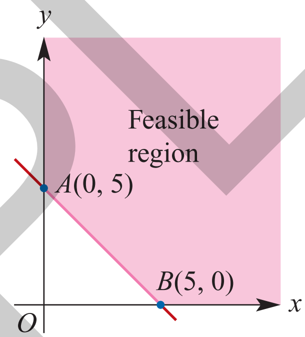
4 $C = x + y$ (minimum)



5 $P = x + 2y$ (maximum)



6 $C = 2x + 2y$ (minimum)



9.5 Linear programming applications

You now have all the technical skills necessary to set up and solve a basic linear programming problem.

Example 9 **Setting up and solving a maximising problem**

A manufacturer makes two sorts of orange-flavoured chocolates: House Brand and Orange Delights.

- 1 kg of House Brand contains 0.3 kg of chocolate and 0.7 kg of orange fill.
- 1 kg of Orange Delights contains 0.5 kg of chocolate and 0.5 kg of orange fill.
- 300 kg of chocolate and 350 kg of orange fill are available to the manufacturer each day.
- The profit is \$7.50 per kilogram on House Brand and \$10 per kilogram on Orange Delights.

How much of each type of orange-flavoured chocolate should be made each day to maximise profit?

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Solution

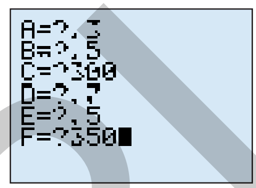
1 Define x and y.

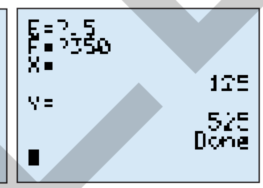
Let x be the amount (in kg) of House Brand made each day.
Let y be the amount (in kg) of Orange Delights made each day.

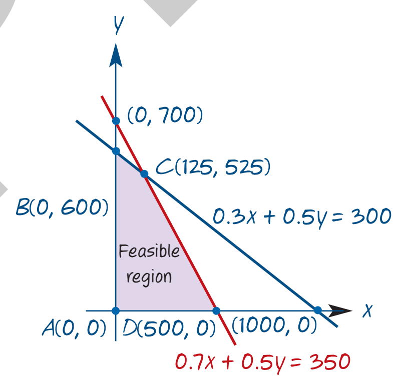
2 Write down the constraints.
- x and y cannot be negative.
- 300 kg of chocolate is available.
- 350 kg of orange fill is available.

Constraints:
$x \geq 0, \quad y \geq 0$
$0.3x + 0.5y \leq 300$ (chocolate)
$0.7x + 0.5y \leq 350$ (orange fill)

3 Graph the feasible region defined by the constraints. Mark in each of the corner points and label with their coordinates. Use a calculator to determine the point of intersection.







4 Write down the objective function (in dollars). Call it P, for profit.

Objective function:
$P = 7.5x + 10y$

5 Determine the maximum profit by evaluating the objective function at each corner of the feasible region.

Point	Objective function $P = 7.5x + 10y$
A(0, 0)	$P = 7.5 \times 0 + 10 \times 0 = \$0$
B(0, 600)	$P = 7.5 \times 0 + 10 \times 600 = \$6000$
C(125, 525)	$P = 7.5 \times 125 + 10 \times 525 = \$6187.50$
D(500, 0)	$P = 7.5 \times 500 + 10 \times 0 = \$3750$

6 Write your answer to the question.

The maximum profit is \$6187.50, which is obtained by making 125 kg of House Brand and 525 kg of Orange Delights.

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Example 10 Setting up and solving a minimising problem

SpeedGro and Powerfeed are two popular brands of home garden fertiliser. They both contain the nutrients X, Y and Z, needed for healthy plant growth.

- 1 kg of SpeedGro contains 30 units of X, 50 units of Y and 10 units of Z.
- 1 kg of Powerfeed contains 20 units of X, 20 units of Y and 20 units of Z.
- A gardener calculates that he needs a fertiliser containing at least 160 units of nutrient X, 200 units of nutrient Y and 80 units of nutrient Z.
- Speedgro costs \$8 per kg and Powerfeed costs \$6 per kg.

How much of each type of fertiliser should he buy to meet his needs at the minimum cost?

Solution

1 Define x and y.

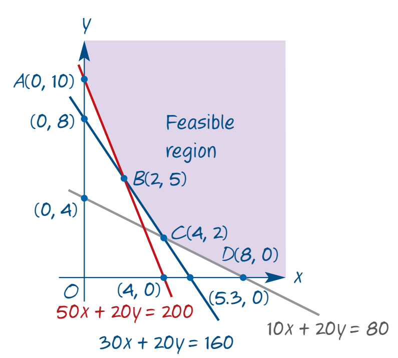
Let x be the amount (in kg) of SpeedGro needed.
Let y be the amount (in kg) of Powerfeed needed.

2 Write down the constraints.

- x and y cannot be negative.
- At least 160 units of X are needed.
- At least 200 units of Y are needed.
- At least 80 units of Z are needed.

Constraints:
$x \geq 0, \quad y \geq 0$
$30x + 20y \geq 160$ (nutrient X)
$50x + 20y \geq 200$ (nutrient Y)
$10x + 20y \geq 80$ (nutrient Z)

3 Graph the feasible region defined by the constraints. Mark in each of the corner points and label with their coordinates. Use a calculator to determine the points of intersection.



4 Write down the objective function (in dollars). Call it C, for cost.

Objective function:
$C = 8x + 6y$

5 Determine the minimum cost by evaluating the objective function at each corner of the feasible region.

Point	Objective function $C = 8x + 6y$
A(0, 10)	$C = 8 \times 0 + 6 \times 10 = \$60$
B(2, 5)	$C = 8 \times 2 + 6 \times 5 = \$46$
C(4, 2)	$C = 8 \times 4 + 6 \times 2 = \$44$
D(8, 0)	$C = 8 \times 8 + 6 \times 0 = \$64$

6 Write your answer to the question.

The minimum cost is \$44, which is achieved by buying 4 kg of SpeedGro and 2 kg of Powerfeed.

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Exercise 9E

1 A factory makes two products: Wigits and Gigits. Two different machines are used.

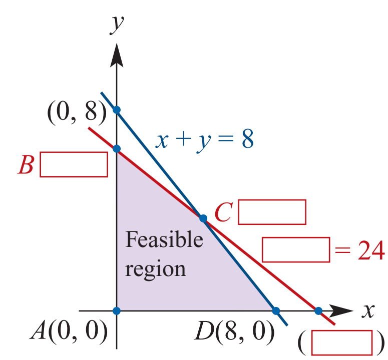
- To make a Wigit takes 1 hour on Machine 1 and 2 hours on Machine 2.
- To make a Gigit takes 1 hour on Machine 1 and 4 hours on Machine 2.
- Up to 8 hours of Machine 1 time and up to 24 hours of Machine 2 time are available each day.
- The factory makes a profit of $200 for each Wigit and $360 for each Gigit it produces.

a Let x be the number of Wigits made each day.
Let y be the number of Gigits made each day.
The constraints for this problem are:

$$x \geq 0, \quad y \geq \square$$
$$x + y \leq 8 \quad \text{(Machine 1 time)}$$
$$\square x + 4y \leq \square \quad \text{(Machine 2 time)}$$

Determine the missing information.

b The feasible region is shown on the right. Some information is missing. Determine the missing information.



c The objective function is give by $P = 200x + \square y$, where P stands for profit (in dollars). Determine the missing information.

d How many Wigits and Gigits should be made each day to maximise profit, and what is this profit?

2 An outdoor clothing manufacturer makes two sorts of jackets: Polarbear and Polarfox.

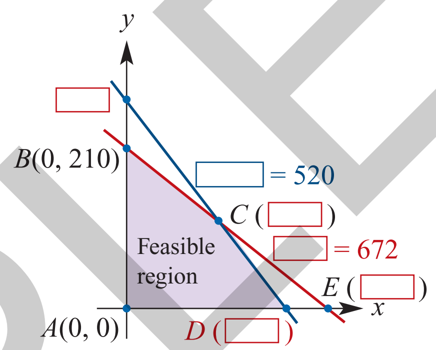
- To make a Polarbear jacket takes 2 m of material. The time taken to make a Polarbear jacket is 2.4 hours.
- To make a Polarfox jacket takes 2 m of material. The time taken to make a Polarfox jacket is 3.2 hours.
- The manufacturer has 520 m of material available and 672 hours of worker time to make the jackets.
- The manufacturer makes a profit of $36 for each Polarbear jacket and $42 for each Polarfox jacket it produces.

a Let x be the number of Polarbear jackets made.
Let y be the number of Polarfox jackets made.
The constraints for this problem are:

$$x \geq \square, \quad y \geq \square$$
$$\square x + 2y \leq \square \quad \text{(material availability)}$$
$$\square x + \square y \leq 672 \quad \text{(worker time availability)}$$

Determine the missing information.

b The feasible region is shown on the right. Some information is missing. Determine the missing information.



c The objective function is given by $P = \square x + \square y$, where P stands for profit (in dollars)
Determine the missing information.

d What is the maximum profit that can be made, and how many Polarbear jackets and Polarfox jackets should be made each day to achieve this profit?

3 Following a natural disaster, the army plans to use helicopters to transport medical teams and their equipment into a remote area. They have two types of helicopter: Redhawks and Blackjets.

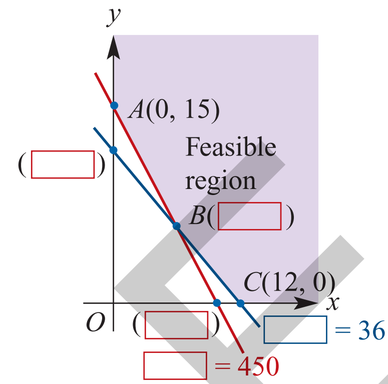
- Redhawks carry 45 people and 3 tonnes of equipment.
- Blackjets carry 30 people and 4 tonnes of equipment.
- At least 450 people and 36 tonnes of equipment need to be transported.
- Redhawks cost \$3600 per hour to run and Blackjets cost \$3200 per hour to run.

a Let x be the number of Redhawks.
Let y be the number of Blackjets.
The constraints for this problem are:

$$x \geq 0, \quad y \geq 0$$
$$\square \quad \text{(people)}$$
$$\square \quad \text{(equipment)}$$

Determine the missing information.

b The feasible region is shown on the right. Some information is missing. Determine the missing information.



c The objective function is given by $C = \square x + \square y$, where C stands for cost (in dollars). Determine the missing information.

d How many Redhawks and Blackjets should be used to minimise the cost per hour, and what is this cost?

4 A sawmill produces both construction grade and furniture grade timber.

- To produce 1 cubic metre of construction grade timber takes 2 hours of sawing and 3 hours of planing.
- To produce 1 cubic metre of furniture grade timber takes 2 hours of sawing and 6 hours of planing.
- Up to 8 hours of sawing time and 18 hours of planing time are available each day.
- The sawmill makes a profit of $500 per cubic metre of construction grade timber and $600 per cubic metre of furniture grade timber it produces.

a Write down the constraints and profit function for this problem.

b Draw a diagram.

c Find how much construction grade and furniture grade timber the sawmill should make each day to maximise its profit. What is this profit?

5 Two breakfast cereal mixes, Healthystart and Wakeup, are available in bulk.

- Each kilogram of Healthystart contains 12 mg of vitamin B1 and 40 mg of vitamin B2.
- Each kilogram of Wakeup contains 20 mg of vitamin B1 and 25 mg of vitamin B2.
- You want a mix of the two that contains at least 15 mg of vitamin B1 and 30 mg of vitamin B2.
- Healthystart costs $5 a kilogram and Wakeup costs $4.50 per kilogram.

a Write down the constraints and cost function for this problem.

b Draw a diagram.

c Find the mixture of these two cereals that will meet your needs at minimum cost. What is this cost?

MC TEST D&D TEST

Key ideas and chapter summary

Linear inequality	A **linear inequality** involves one or two of the signs $>$, $\geq$, $<$ or $\leq$, but *not* an equals sign ($=$).
Displaying linear inequalities in one variable on a number line	A linear inequality in one variable can be represented on a number line by a solid coloured line ending at one or two circles. The line represents all the possible solutions of the inequality. An **open circle** (○) indicates that the end value is *not* included in the inequality (for $<$ or $>$). A **closed circle** (●) indicates that the end value is included in the inequality (for $\leq$ or $\geq$).
Displaying linear inequalities in one variable on the coordinate plane	Linear inequalities in one variable can be represented on a coordinate plane by a shaded region bounded by one or two lines parallel to the x- or y-axes. The region represents all the possible solutions of the inequality. **A dashed line** indicates that the line is *not* included in the inequality (for $<$ or $>$). **A solid line** indicates that the line is included in the inequality (for $\leq$ or $\geq$).
Displaying linear inequalities in two variables on the coordinate plane	A linear inequality in two variables can be represented on a coordinate plane by a shaded region bounded by a line at an angle to the x- and y-axes. The region represents all the possible solutions of the inequality. The boundary line is **dashed** if it is *not* included in the inequality (for $<$ or $>$), but **solid** if it is included (for $\leq$ or $\geq$). A **reference point**, often the origin $(0, 0)$, can be used to help decide whether the required region lies above or below the line.
Feasible region	When solving simultaneous inequalities, the region in the coordinate plane that is common to all the inequalities is called the **feasible region**. It represents all the possible solutions to the simultaneous inequalities. The feasible region can be found graphically (for a small number of inequalities) by **shading in** the required regions for all the inequalities and determining where they all overlap. A graphics calculator can be used to graph a feasible region.
Linear programming	**Linear programming** involves **maximising** or **minimising** a linear quantity subject to the **constraints** represented by a set of linear inequalities. The constraints (e.g. requirements, resources) define the feasible region in which the quantity is to be maximised or minimised.

	The constraints $x \geq 0$ and $y \geq 0$ together restrict the feasible region to the positive (first) quadrant.
Objective function	The **objective function** is a linear expression representing the quantity to be maximised (e.g. profit) or minimised (e.g. cost) in a linear programming problem.
Corner point principle	The **corner point principle** states that, in linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region, or on a line on the boundary of the feasible region joining two of the corners.

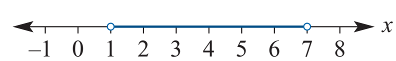
Skills check

Having completed this topic you should be able to:

- represent a linear inequality in one variable on a number line
- represent a linear inequality in one or two variables on the coordinate plane
- know the meaning of the terms feasible region, constraint and objective function as they relate to linear programming
- determine the maximum or minimum value of an objective function for a given feasible region
- set up and solve basic linear programming problems.

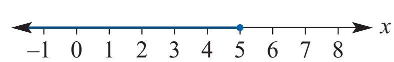
Multiple-choice questions

1 The inequality displayed on the number line on the right is:



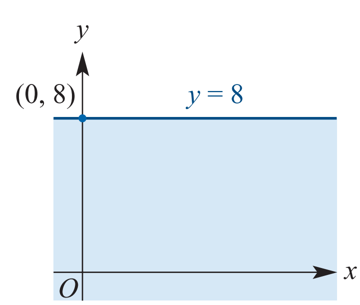
A $1 \leq x \leq 7$ **B** $1 < x < 7$ **C** $1 \leq x < 7$
D $1 < x \leq 7$ **E** $1 > x > 7$

2 The inequality displayed on the number line on the right is:



A $x < 5$ **B** $x \leq 5$ **C** $x > 5$
D $x \geq 5$ **E** $0 > x > 5$

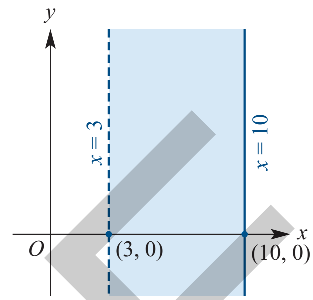
3 The inequality displayed on the coordinate plane on the right is:



A $x < 8$ **B** $x \leq 8$ **C** $y < 8$
D $y \leq 8$ **E** $0 > x > 8$

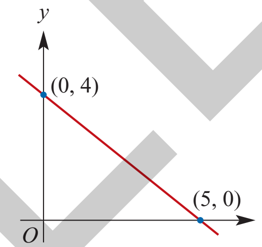
4 The inequality displayed on the coordinate plane on the right is:

A $3 < x < 10$ **B** $3 < x \leq 10$ **C** $3 \leq x \leq 10$
D $3 < y < 10$ **E** $3 < y \leq 10$



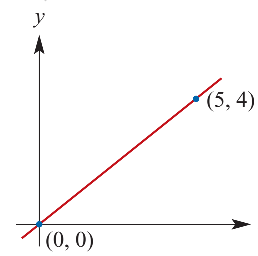
5 The equation of the line displayed on the right is:

A $4x + 5y = 4$ **B** $4x - 5y = 4$ **C** $5x + 4y = 20$
D $4x + 5y = 20$ **E** $4x - 5y = 20$



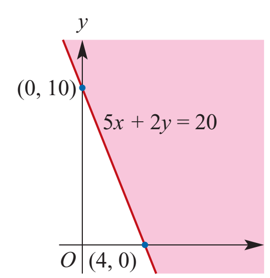
6 The equation of the line displayed on the right is:

A $4x - 5y = 0$ **B** $4x + 5y = 0$ **C** $5x - 4y = 20$
D $5x + 4y = 20$ **E** $5y = 20x$



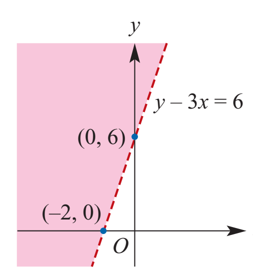
7 The region displayed on the right (including the line) represents the inequality:

A $5x + 2y < 20$ **B** $5x + 2y \leq 20$ **C** $5x + 2y > 20$
D $5x + 2y \geq 20$ **E** $2x + 5y > 20$



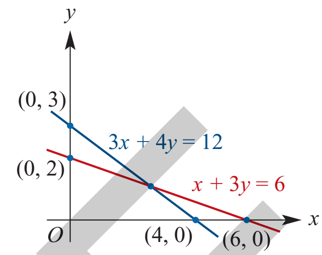
8 The region displayed on the right (not including the line) represents the inequality:

A $y - 3x \leq 6$ **B** $y - 3x < 6$ **C** $y - 3x \geq 6$
D $y - 3x > 6$ **E** $3x - y > 6$



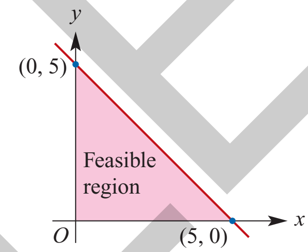
9 The two lines shown on the right intersect at the point:

A (1, 1.3) **B** (2, 1.5) **C** (1.2, 1.2)
D (2.4, 1.2) **E** (3, 2.4)



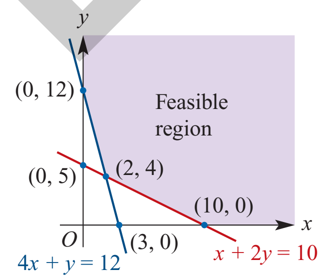
10 The feasible region displayed on the right (including the line) is defined by the inequalities:

A $x \geq 0, \quad y \geq 0, \quad x - y < 5$
B $x \geq 0, \quad y \geq 0, \quad x - y \geq 5$
C $x \geq 0, \quad y \geq 0, \quad x + y < 5$
D $x \geq 0, \quad y \geq 0, \quad x + y \leq 5$
E $x \geq 0, \quad y \geq 0, \quad x + y \geq 5$



11 The feasible region displayed on the right (including the lines) is defined by the inequalities:

A $x \geq 0, \quad y \geq 0, \quad x + 2y \geq 10, \quad 4x + y \geq 12$
B $x \geq 0, \quad y \geq 0, \quad x + 2y \leq 10, \quad 4x + y \leq 12$
C $x \geq 0, \quad y \geq 0, \quad x + 2y > 10, \quad 4x + y > 12$
D $x \geq 0, \quad y \geq 0, \quad x + 2y < 10, \quad 4x + y < 12$
E $x \geq 0, \quad y \geq 0, \quad x + 2y \geq 10, \quad 4x + y \leq 12$

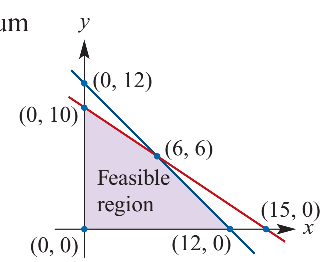


12 For the feasible region displayed in Question **11**, the minimum value of the objective function, $C = 2x + y$, is:

A 5 **B** 6 **C** 8 **D** 12 **E** 20

13 For the feasible region displayed on the right, the maximum value of the objective function, $P = 4x + 3y$, is:

A 0 **B** 40 **C** 42
D 48 **E** 60



The following information relates to Questions 14 to 16

An outdoor clothing manufacturer makes two styles of all-weather coats: long and short.

- To make a short coat, 2 m of material are required. The time taken to make a short coat is 2.5 hours.
- To make a long coat, 3 m of material are required. The time taken to make a long coat is 3.5 hours.

- The manufacturer has 450 m of material available and 700 hours of worker time to make the coats.
- The manufacturer makes a profit of \$40 for each short coat and \$48 for each long coat.

Let x be the number of short coats made.
Let y be the number of long coats made.

14 The constraints that relate to the amount of *material* available are:

A $x \geq 0,\quad y \geq 0,\quad 2x + 3y \leq 450$
B $x \geq 0,\quad y \geq 0,\quad 2x + 3y \geq 450$
C $x \geq 0,\quad y \geq 0,\quad 2.5x + 3.5y \leq 700$
D $x \geq 0,\quad y \geq 0,\quad 2.5x + 3.5y \geq 700$
E $x \geq 0,\quad y \geq 0,\quad 40x + 48y \geq 700$

15 The constraints that relate to the amount of *time* available are:

A $x \geq 0,\quad y \geq 0,\quad 2x + 3y \leq 450$
B $x \geq 0,\quad y \geq 0,\quad 2x + 3y \geq 450$
C $x \geq 0,\quad y \geq 0,\quad 2.5x + 3.5y \leq 700$
D $x \geq 0,\quad y \geq 0,\quad 2.5x + 3.5y \geq 700$
E $x \geq 0,\quad y \geq 0,\quad 40x + 48y \geq 700$

16 The objective function P is:

A $P = 2.5x + 3.5y$
B $P = 2x + 3y$
C $P = 3x + 3.5y$
D $P = 40x + 48y$
E $P = 450x + 700y$

Short-answer questions

1 Plot the inequality $-2 \leq x < 4$ on a number line.

2 Plot the inequality $1 \leq y < 5$ on the coordinate plane.

3 Plot the inequality $5x + 4y < 40$ on the coordinate plane.

4 Plot the region defined by the inequalities:

$$x \geq 0, \quad y \geq 0, \quad 3x + 5y \leq 60$$

5 Plot the region defined by the inequalities:

$$x \geq 0, \quad y \geq 0, \quad 2x + 3y \geq 30, \quad x + 4y \geq 20$$

Extended-response questions

1 A garden products company makes two sorts of fertiliser: Standard Grade and Premium Grade. There are two main ingredients: nitrate and phosphate.

- To make a tonne of Standard Grade fertiliser takes 0.8 tonnes of nitrate and 0.2 tonnes of phosphate.
- To make a tonne of Premium Grade fertiliser takes 0.7 tonnes of nitrate and 0.3 tonnes of phosphate.
- The company has 56 tonnes of nitrate and 21 tonnes of phosphate.
- The company makes a profit of \$600 per tonne on Standard Grade fertiliser and \$750 per tonne on Premium Grade fertiliser.

Review

a Write the constraints and profit function for this problem.
b Draw a diagram.
c Find how much of each type of fertiliser the company should make to maximise its profit. What will this profit be?

2 Two foods fed to animals contain both vitamin A and vitamin B.

- 1 kg of Food A contains 3 units of vitamin A and 4 units of vitamin B.
- 1 kg of Food B contains 5 units of vitamin A and 3 units of vitamin B.
- The daily vitamin requirement of each animal is at least 15 units of vitamin A and at least 12 units of vitamin B.
- Food A costs $0.30 per kg and Food B costs $0.24 per kg.

a Write the constraints and cost function for this problem.
b Draw a diagram.
c Find how much of each type of food should be fed to the animals each day to minimise cost. What is this cost?