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CHAPTER

7

Trigonometry

- How are sin θ, cos θ and tan θ defined using a right-angled triangle?
- How can the trigonometric ratios be used to find the side lengths or angles in right-angled triangles?
- What is meant by an angle of elevation or an angle of depression?
- How are compass bearings and true bearings measured?
- How can the sine and cosine rules be used to solve non-right-angled triangles?
- What are the three rules that can be used to find the area of a triangle?

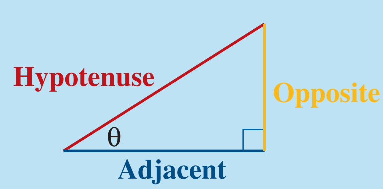
Trigonometry can be used to solve many practical problems. How high is that tree? What is the height of the mountain we can see in the distance? What is the exact location of the fire that has just been seen by fire spotters? How wide is the lake? What is the area of this irregular-shaped paddock?

7.1 Trigonometry basics

Although you are likely to have studied some trigonometry, it may be helpful to review a few basic ideas.

Naming the sides of a right-angled triangle

- The **hypotenuse** is the longest side of the right-angled triangle and is always opposite the right angle (90°).
- The **opposite** side is directly opposite the angle θ.
- The **adjacent** side is beside the angle θ, but it is not the hypotenuse. It runs from θ to the right angle.



The **opposite** and **adjacent** sides are located in relation to the position of angle θ. If θ was in the other corner, the sides would have to swap their labels. The letter θ is the Greek letter **theta**. It is commonly used to label an angle.

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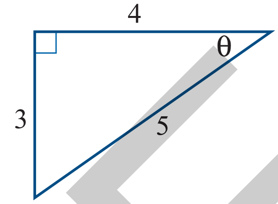
Example 1 **Identifying the sides of a right-angled triangle**

Give the values of the hypotenuse, the opposite side and the adjacent side in the triangle shown.

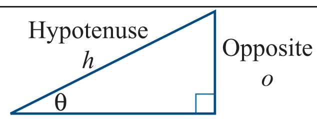
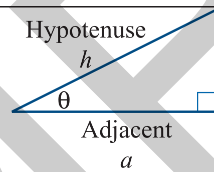
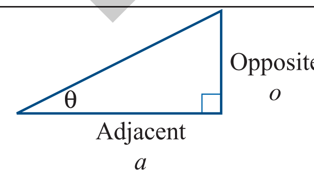


Solution

- The hypotenuse is opposite the right angle. — The hypotenuse, $h = 5$
- The opposite side is opposite the angle θ. — The opposite side, $o = 3$
- The adjacent side is beside θ, but is not the hypotenuse. — The adjacent side, $a = 4$

The trigonometric ratios

The trigonometric ratios **sin θ**, **cos θ** and **tan θ** can be defined in terms of the sides of a right-angled triangle.

		
$\sin\theta = \dfrac{\textbf{opposite}}{\textbf{hypotenuse}}$	$\cos\theta = \dfrac{\textbf{adjacent}}{\textbf{hypotenuse}}$	$\tan\theta = \dfrac{\textbf{opposite}}{\textbf{adjacent}}$
$\sin\theta = \dfrac{o}{h}$	$\cos\theta = \dfrac{a}{h}$	$\tan\theta = \dfrac{o}{a}$

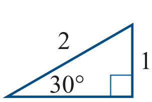
's o h ------ c a h ----- t o a'

This mnemonic is often used by students to help them remember the rule for each trigonometric ratio. Or you may prefer:

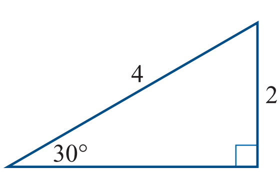
'**S**ir **O**liver's **H**orse **C**ame **A**mbling **H**ome **T**o **O**liver's **A**rms'

The meaning of the trigonometric ratios

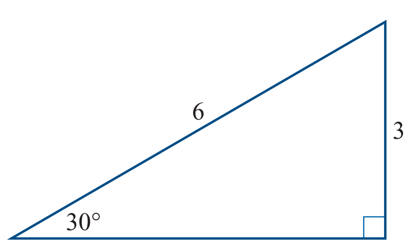
Using a calculator, we find, for example, that sin 30° = 0.5. This means that in *all* right-angled triangles with an angle of 30°, the ratio of the side opposite the 30° to the hypotenuse is always 0.5.



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} = 0.5$$



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{4} = 0.5$$



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{6} = 0.5$$

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Try drawing any right-angled triangle with an angle of 30° and check that the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}} = 0.5$

Similarly, for *any* right-angled triangle with an angle of 30° the ratios cos 30° and tan 30° always have the same values:

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ is always } \frac{\sqrt{3}}{2} = 0.8660 \ \text{(to 4 decimal places)}$$

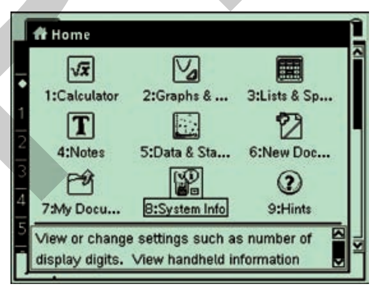
$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} \text{ is always } \frac{1}{\sqrt{3}} = 0.5774 \ \text{(to 4 decimal places).}$$

A calculator gives the value of each trigonometric ratio for any angle entered.

TI-Nspire CAS tip

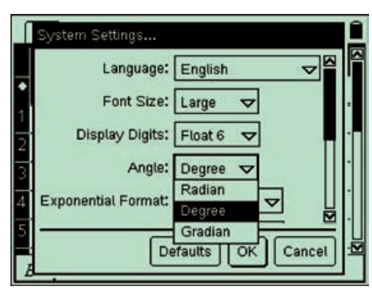
When solving problems in trigonometry, your calculator should be kept in **Degree** mode.

Press (home) **/8:System Info/2:System Settings**.

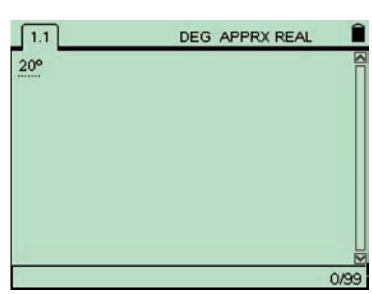


Use the (tab) key to highlight the **Angle** entry box. Press ▼ to access the choices and use ▼ or ▲ arrows to highlight **Degree**. Press (enter).

Press (enter) twice to accept this change.



In addition, it is recommended that you always press (ctrl) + (°) to insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.



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ClassPad tip

When solving problems in trigonometry, your calculator should be kept in **Degree** mode.

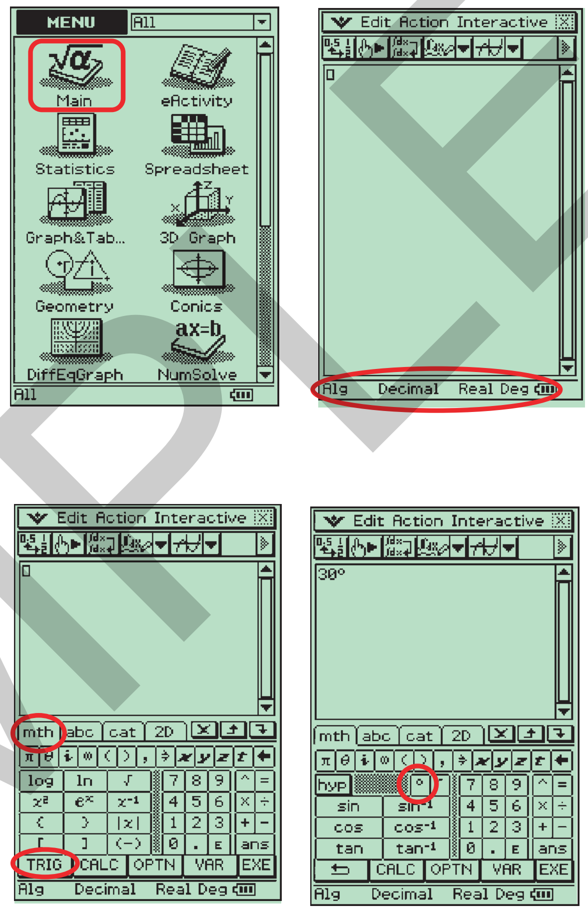
Open the **Main** application.
The status line at the bottom of the application screen is used to set your calculator to work with angles in degrees and to display answers as decimals.

The settings you require are, reading from the left:
Alg, Decimal, Real and **Degree**.
If **Standard** not **Decimal** shows, tap to change. If **Gra** or **Rad**, not **Deg** show, tap to change.

In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any calculator settings and reaffirms an angle measurement, not a length.

To access the degree symbol, press (Keyboard) on the front of the calculator. Tap the mth tab and then the TRIG menu item at the bottom of the keyboard window.

After entering the angle size, tap the degree symbol (°) to insert its symbol.



Example 2 — Finding the values of trigonometric ratios

Use your graphics calculator to find, correct to 4 decimal places, the value of:

a sin 49° **b** cos 16° **c** tan 27.3°

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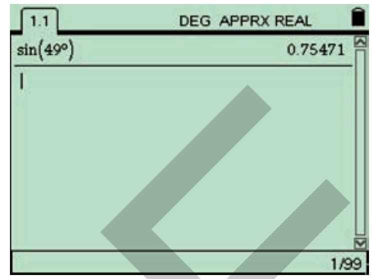
Solution

a On the calculation screen



1 For **TI-nspire CAS**, press sin 4 9 ctrl ° enter.

Notes:

1 Pressing ctrl ° enters the degree sign (°).

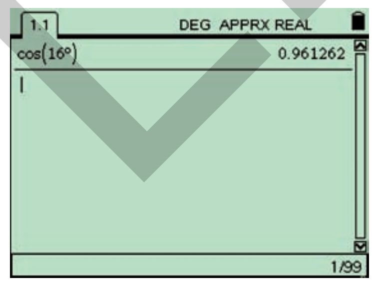
2 If your answer is not a decimal, press ctrl enter. Alternatively, set your calculator to **Approximate** (decimal) mode (see the Appendix).

2 For **Classpad**, display the keyboard (Keyboard), tap the mth tab, then the TRIG menu. To enter and evaluate the expression, tap sin 4 9 °) EXE.

3 Write your answer, correct to 4 decimal places.

$\sin 49^\circ = 0.7547$

b On the calculation screen



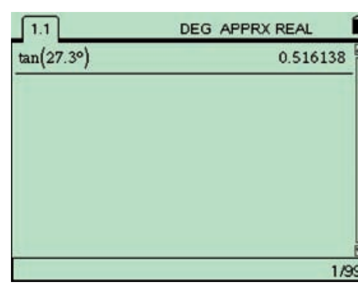
1 For **TI-nspire CAS**, press cos 1 6 ctrl ° enter.

2 For **Classpad**, tap cos 1 6 °) EXE.

3 Write your answer, correct to 4 decimal places.

$\cos 16^\circ = 0.9613$

c On the calculation screen



1 For **TI-nspire CAS**, press tan 2 7 . 3 ctrl ° enter.

2 For **Classpad**, tap tan 2 7 • 3 °) EXE.

3 Write your answer, correct to 4 decimal places.

$\tan 27.3^\circ = 0.5161$

In the following two sections we will see that if an angle and a side are known we can find one of the other sides by using the required trigonometric ratio. If two sides of the right-angled triangle are known we can find one of the angles.

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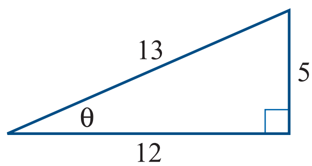
Exercise 7A

1 State the values of the hypotenuse, the opposite side and the adjacent side in each triangle.

a



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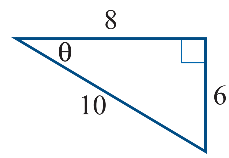

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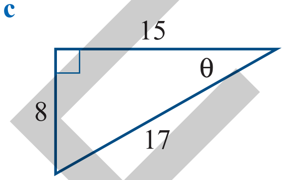

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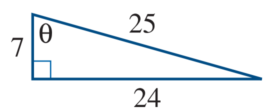

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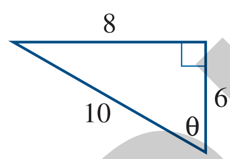

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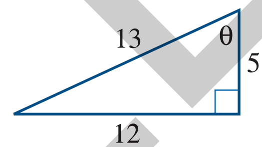


2 Write the ratios for sin θ, cos θ and tan θ for each triangle in Question **1**.

3 Find the values of the following trigonometric ratios, correct to 4 decimal places.

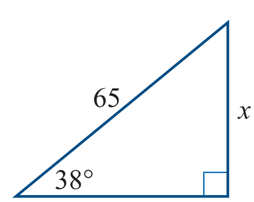
a $\sin 27^\circ$ **b** $\cos 43^\circ$ **c** $\tan 62^\circ$ **d** $\cos 79^\circ$ **e** $\tan 14^\circ$ **f** $\sin 81^\circ$
g $\cos 17^\circ$ **h** $\tan 48^\circ$ **i** $\sin 80^\circ$ **j** $\sin 49.8^\circ$ **k** $\tan 80.2^\circ$ **l** $\cos 85.7^\circ$

7.2 Finding an unknown side in a right-angled triangle

The trigonometric ratios can be used to find unknown sides in a right-angled triangle, given an angle and one side. When the unknown side is in the **numerator** (top) of the trigonometric ratio, proceed as follows.

Example 3 Finding an unknown side

Find the length of the unknown side x in the triangle shown, correct to 2 decimal places.



Solution

1 The sides involved are the opposite and the hypotenuse, so use sin θ.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\sin 38^\circ = \frac{x}{65}$$

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3 Multiply both sides of the equation by 65 to obtain an expression for x. Use a calculator to evaluate.

$$65 \times \sin 38^\circ = x$$
$$x = 65 \times \sin 38^\circ$$
$$= 40.017\ldots$$

4 Write your answer correct to 2 decimal places.

$$x = 40.02$$

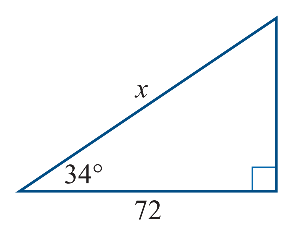
Finding an unknown side in a right-angled triangle

1. Draw the triangle with the given angle and side shown. Label the unknown side as x.
2. Use the trigonometric ratio that includes the given side and the unknown side.
   - If given the opposite and the hypotenuse, use $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
   - If given the adjacent and the hypotenuse, use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
   - If given the opposite and the adjacent, use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
3. Rearrange the equation to make x the subject.
4. Use the appropriate function key to find x.

An extra step is needed when the unknown side is in the **denominator** (at the bottom) of the trigonometric ratio.

Example 4 Finding an unknown side which is in the denominator of the trig ratio

Find the value of x in the triangle shown, correct to 2 decimal places.



Solution

1 The sides involved are the adjacent and the hypotenuse, so use cos θ.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\cos 34^\circ = \frac{72}{x}$$

3 Multiply both sides by x.

$$x \cos 34^\circ = 72$$

4 Divide both sides by cos 34° to obtain an expression for x. Use a calculator to evaluate.

$$x = \frac{72}{\cos 34^\circ}$$
$$= 86.847\ldots$$

5 Write your answer correct to 2 decimal places.

$$x = 86.85$$

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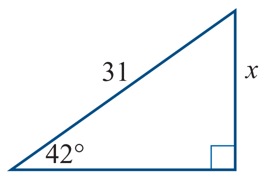
Exercise 7B

1 In each right-angled triangle below, find the unknown side x, correct to 2 decimal places.

a



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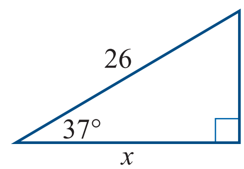

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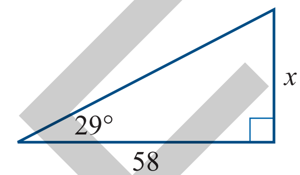

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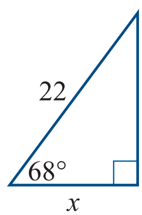

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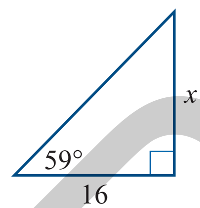

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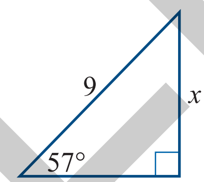

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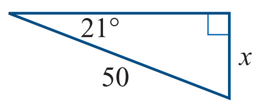

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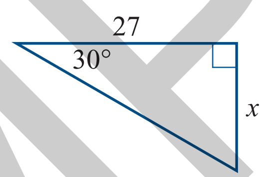

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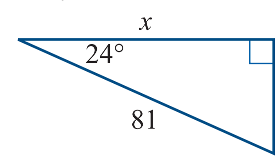

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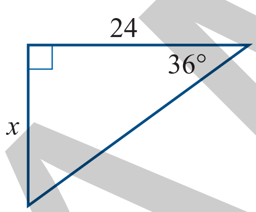

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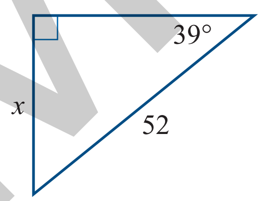

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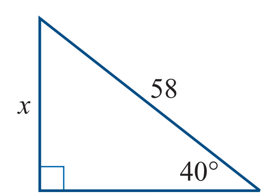
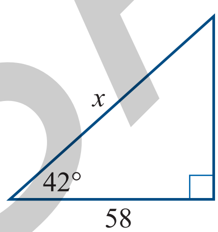


2 Find the unknown side x in each right-angled triangle below, correct to 2 decimal places.

a



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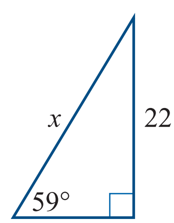

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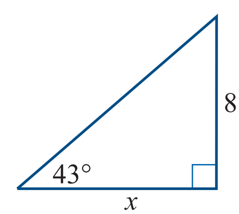

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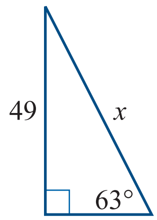
d



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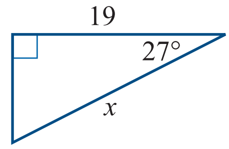

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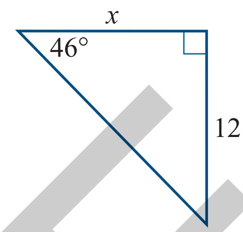

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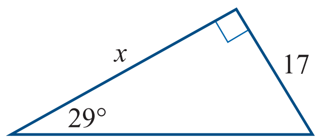

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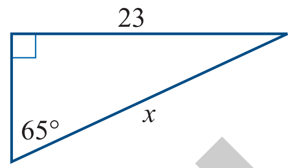

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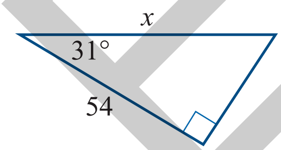

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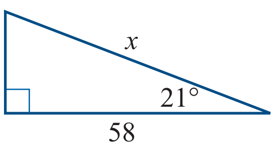

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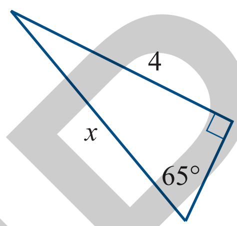

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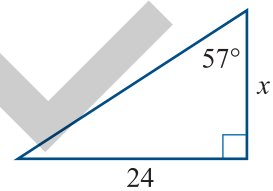
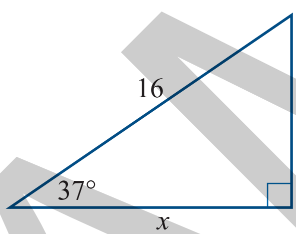


3 Find the length of the unknown side shown in each triangle, correct to 1 decimal place.

a



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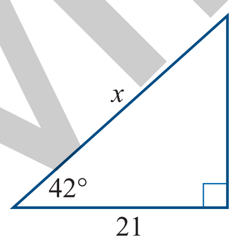

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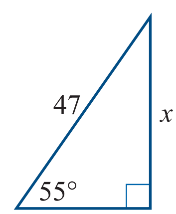

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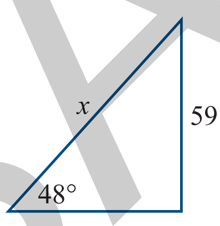

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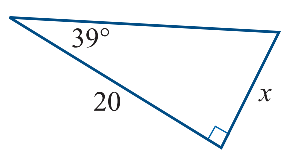

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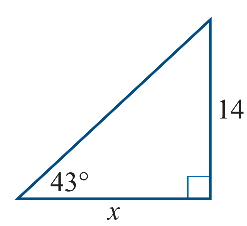

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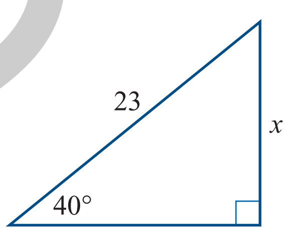

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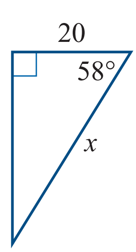

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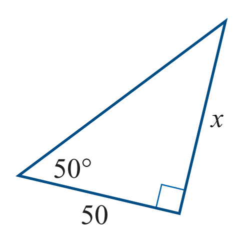

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7.3 Finding an angle in a right-angled triangle

Warning!!

Make sure that your calculator is set in **DEGREE** mode before attempting this section.

Finding an angle from a trigonometric ratio value

Before we look at how to find an unknown angle in a right-angled triangle, it will be useful to see how to find the angle when we know the value of the trigonometric ratio. If we are asked to find θ when

$$\sin \theta = 0.8480$$

it is as if we have to find reverse gear to undo the effect of the SIN key (or button), so that we can go back to see the angle that was used when SIN was pressed (or tapped) to get 0.8480. The reverse gear for the SIN key (or button) is called the **inverse of sine**, written $\mathbf{sin^{-1}}$. The superscript –1 is *not* a power. It's just saying let's undo, or take one step backwards from, applying the sine function. The request to find θ when sin θ = 0.8480 can be written as

$$\sin^{-1} 0.8480 = \theta$$

In the following example we will see how to find θ when sin θ = 0.8480.

Similarly, the **inverse of cosine** is written $\mathbf{cos^{-1}}$, and the **inverse of tangent** is written $\mathbf{tan^{-1}}$.

Example 5 Finding an angle from a trigonometric ratio

Find the angle θ, correct to 1 decimal place, given:

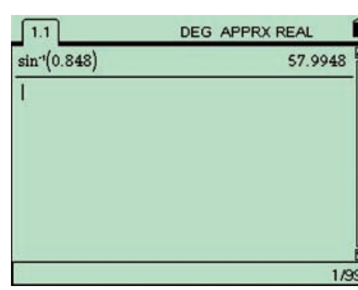
a $\sin \theta = 0.8480$ **b** $\cos \theta = 0.5$ **c** $\tan \theta = 1.67$

Solution

a We need to find $\sin^{-1}(0.8480)$.

1 For **TI-nspire CAS**, press ctrl sin 0 . 8 4 8 0 enter.

2 For **Classpad**, tap sin⁻¹ 0 . 8 4 8 0) EXE.

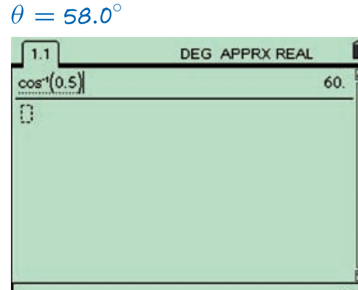


3 Write your answer, correct to 1 decimal place $\theta = 58.0°$

b We need to find $\cos^{-1}(0.5)$.

1 For **TI-nspire CAS**, press ctrl cos 0 . 5 enter.

2 For **Classpad**, tap cos⁻¹ 0 . 5) EXE.



3 Write your answer, correct to 1 decimal place. $\theta = 60°$

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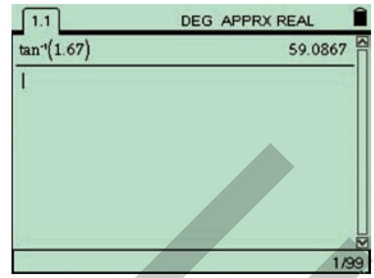
c We need to find $\tan^{-1}(1.67)$.

1 For **TI-nspire CAS**, press ctrl tan 1 . 6 7 enter.

2 For **Classpad**, tap tan 1 • 6 7) EXE.



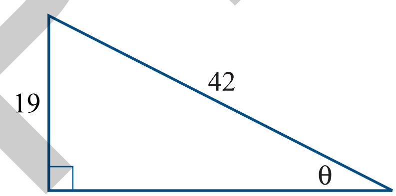
3 Write your answer, correct to 1 decimal place. $\theta = 59.1°$

We can think of the results in Example 5 as follows:

- For '$\sin^{-1} 0.8480 = 58°$', think 'the angle whose sine is 0.8480 equals 58°'.
- For '$\cos^{-1} 0.5 = 60°$', think 'the angle whose cosine is 0.5 equals 60°'.
- For '$\tan^{-1} 1.67 = 59.1°$', think 'the angle whose tangent is 1.67 equals 59.1°'.

Example 6 Finding an angle given two sides in a right-angled triangle

Find the angle θ, in the right-angled triangle shown, correct to 1 decimal place.



Solution

1 The sides involved are the opposite and the hypotenuse, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\sin\theta = \frac{19}{42}$$

3 Write the equation to find an expression for θ. Use a calculator to evaluate.

$$\theta = \sin^{-1}\left(\frac{19}{42}\right) = 26.896\ldots$$

4 Write your answer correct to 1 decimal place.

$$\theta = 26.9°$$

The three angles in a triangle add to 180°. As the right angle is 90°, the other two angles must add to make up the remaining 90°. When one angle has been found, just subtract it from 90° to find the other angle. In Example 6, the other angle must be $90° - 26.9° = 63.1°$.

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Finding an angle in a right-angled triangle

1 Draw the triangle with the given sides shown. Label the unknown angle as θ.

2 Use the trigonometric ratio that includes the two known sides.

- If given the opposite and hypotenuse, use $\sin\ \theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- If given the adjacent and hypotenuse, use $\cos\ \theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- If given the opposite and adjacent, use $\tan\ \theta = \dfrac{\text{opposite}}{\text{adjacent}}$

3 Divide the side lengths to find the value of the trigonometric ratio.

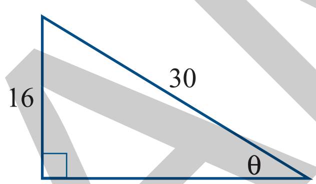
4 Use the appropriate inverse function key to find the angle θ.

Exercise 7C

1 Find the angle θ, correct to 1 decimal place.

a $\sin\theta = 0.4817$ b $\cos\theta = 0.6275$ c $\tan\theta = 0.8666$ d $\sin\theta = 0.5000$
e $\tan\theta = 1.0000$ f $\cos\theta = 0.7071$ g $\sin\theta = 0.8660$ h $\tan\theta = 2.5000$
i $\cos\theta = 0.8383$ j $\sin\theta = 0.9564$ k $\cos\theta = 0.9564$ l $\tan\theta = 0.5774$
m $\sin\theta = 0.7071$ n $\tan\theta = 0.5000$ o $\cos\theta = 0.8660$ p $\cos\theta = 0.3414$

2 Find the unknown angle θ in each triangle, correct to 1 decimal place.

a


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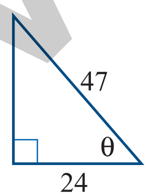

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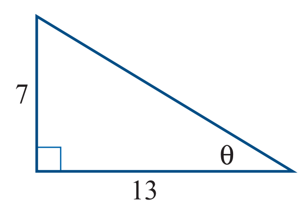
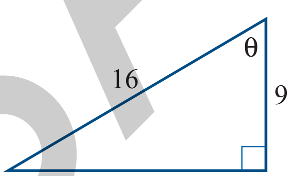


d


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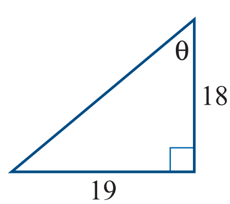

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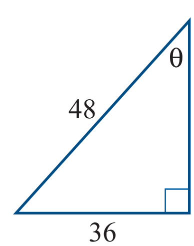

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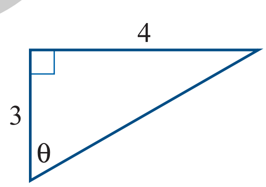

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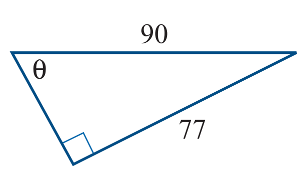

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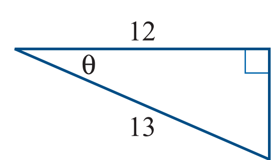

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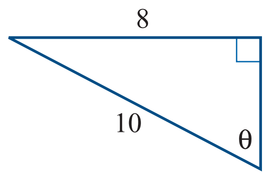
j



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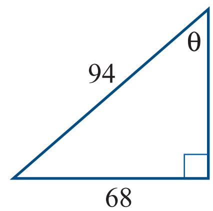

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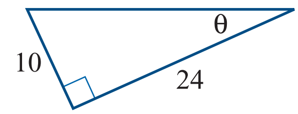

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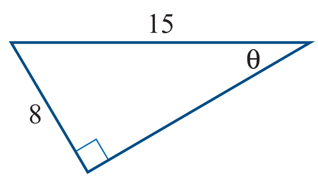

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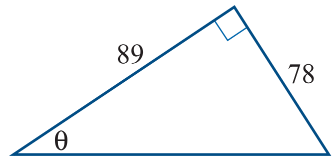
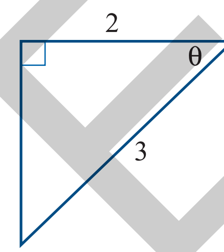
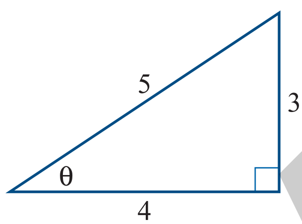


o



3 Find the value of θ in each triangle, correct to 1 decimal place.

a



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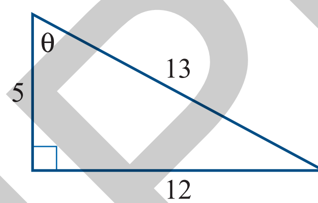

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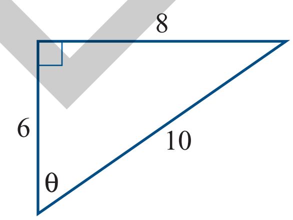

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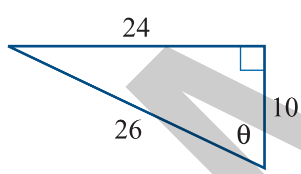

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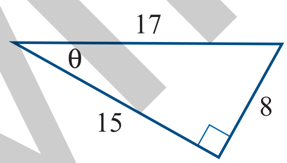

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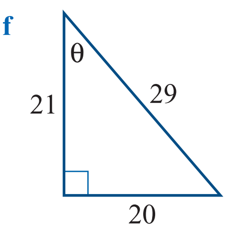
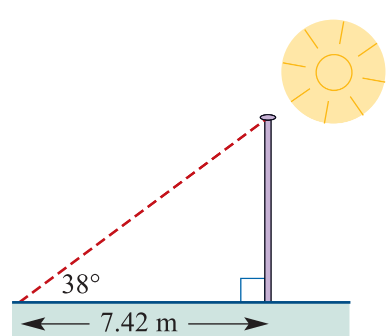


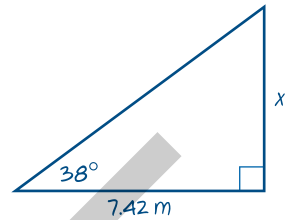
7.4 Applications of right-angled triangles

Example 7 **Application requiring a length**

A flagpole casts a shadow 7.42 m long. The sun's rays make an angle of 38° with the level ground. Find the height of the flagpole, correct to 2 decimal places.



Solution



1 Draw a diagram showing the right-angled triangle. Include all the known details and label the unknown side.

2 The opposite and adjacent sides are involved, so use tan θ.
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

3 Substitute in the known values.
$$\tan 38^\circ = \frac{x}{7.42}$$

4 Multiply both sides by 7.42.
$$7.42 \times \tan 38^\circ = x$$

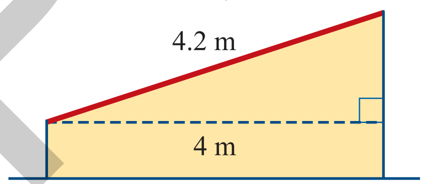
5 Use your calculator to find the value of x.
$$x = 5.797\ldots$$

6 Write your answer correct to 2 decimal places.

The height of the flagpole is 5.80 m.

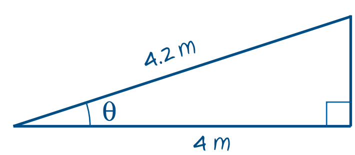
Example 8 Application requiring an angle

A sloping roof uses sheets of corrugated iron 4.2 m long on a shed 4 m wide. There is no overlap of the roof past the sides of the walls. Find the angle the roof makes with the horizontal, correct to 1 decimal place.



Solution

1 Draw a diagram showing the right-angled triangle. Include all known details and label the required angle.



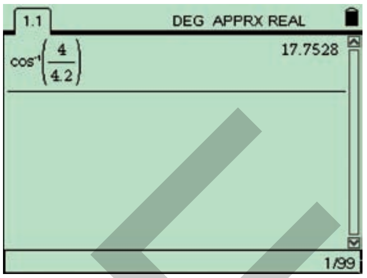
2 The adjacent and hypotenuse are involved, so use cos θ.
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

3 Substitute in the known values.
$$\cos\theta = \frac{4}{4.2}$$

4 Write the equation to find θ.
$$\theta = \cos^{-1}\left(\frac{4}{4.2}\right)$$

5 Use your calculator to find the value of θ.

$\theta = 17.752\ldots$



6 Write your answer correct to 1 decimal place.

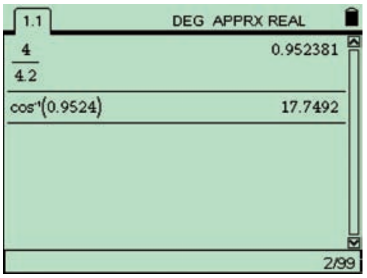
The roof makes an angle of 17.8° with the horizontal.

Warning!!

Always evaluate a mathematical expression as a whole, rather than breaking it into several smaller calculations. Rounding-off errors accumulate as more approximate answers are fed into the calculations.

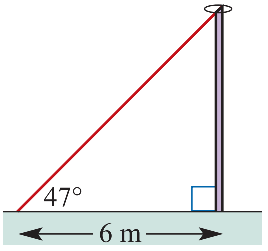
Surprisingly, a value of the trigonometric ratio correct to 4 decimal places can still give an angle that is *not* correct to 3 decimal places!

In Example 8, if we used the value of $\frac{4}{4.2}$ correct to 4 decimal places (0.9524), the angle obtained (17.749) would not even be correct to 3 decimal place (17.753°).

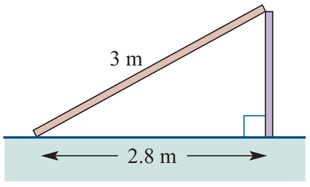


Exercise 7D

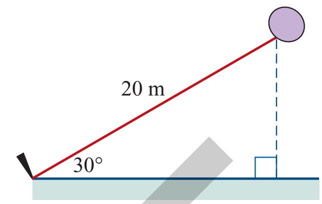
1 A pole is supported by a wire that runs from the top of the pole to a point on the level ground 6 m from the base of the pole. The wire makes an angle of 47° with the ground. Find the height of the pole, correct to 2 decimal places.



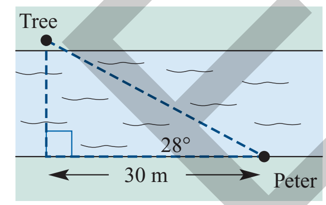
2 A 3 m log rests with one end on the top of a post and the other end on the level ground 2.8 m from the base of the post. Find the angle the log makes with the ground, correct to 1 decimal place.



3 A balloon is tied to a string 20 m long. The other end of the string is secured by a peg to the surface of a level sports field. The wind blows so that the string forms a straight line making an angle of 30° with the ground. Find the height of the balloon above the ground.



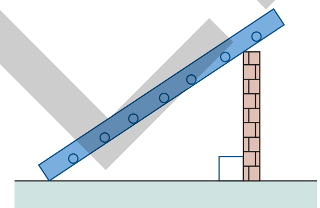
4 Peter noticed that a tree was directly opposite him on the far bank of the river. After he walked 30 m along his side of the river, he found that his line of sight to the tree made an angle of 28° with the river bank. Find the width of the river, to the nearest metre.



5 A ladder rests on a wall 2 m high. The foot of the ladder is 3 m from the base of the wall on level ground.

a Copy the diagram and include the given information. Label as θ the angle the ladder makes with the ground.

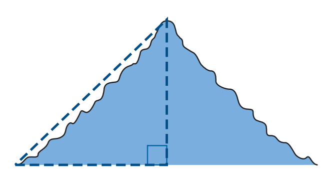
b Find the angle the ladder makes with the ground, correct to 1 decimal place.



6 The distance measured up the sloping face of a mountain was 3.8 km. The sloping face was at an angle of 52° with the horizontal.

a Make a copy of the diagram and show the known details. Show the height of the mountain as x.

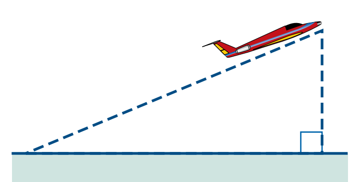
b Find the height of the mountain, correct to 1 decimal place.



7 An aeroplane maintains a flight path of 17° with the horizontal after it takes off. It travels for 2 km along that flight path.

a Show the given and required information on a copy of the diagram.

b Find, correct to 2 decimal places, the horizontal distance of the aeroplane from its take-off point and the height of the aeroplane above ground level.



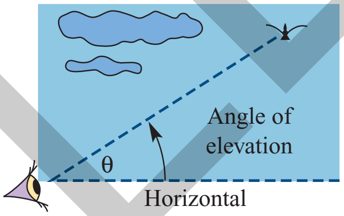
8 A 3 m ladder rests against an internal wall. The foot of the ladder is 1 m from the base of the wall. Find the angle the ladder makes with the floor, correct to 1 decimal place.

9 The entrance to a horizontal mining tunnel has collapsed, trapping the miners inside. The rescue team decide to drill a vertical escape shaft from a position 200 m further up the hill. If the hill slopes at 23° from the horizontal, how deep does the rescue shaft need to be to meet the horizontal tunnel? Answer correct to 1 decimal place.

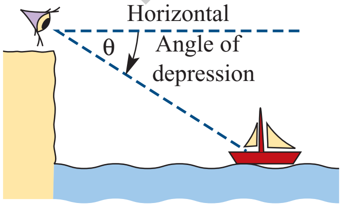
10 A strong rope needs to be fixed with one end attached to the top of a 5 m pole and the other end pegged at an angle of 60° with the level ground. Find the required length of the rope, correct to 2 decimal places.

7.5 Angles of elevation and depression

The **angle of elevation** is the angle through which you *raise* your line of sight from the horizontal when you are looking *up* at something.

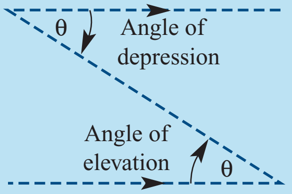


The **angle of depression** is the angle through which you *lower* your line of sight from the horizontal when you are looking *down* at something.



Angle of elevation = angle of depression

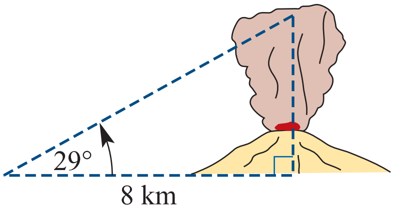
The diagram shows that the angle of elevation and the angle of depression are alternate angles (‘Z’ angles), so they are equal.



Applications of angles of elevation and depression

Example 9 **Angle of elevation**

A park ranger measured the top of a plume of volcanic ash to be at an angle of elevation of 29°. From her map she noted that the volcano was 8 km away. Show how she calculated the height above level ground of the plume of volcanic ash, correct to 1 decimal place.



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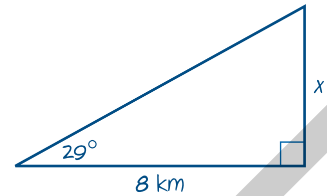
Solution

1 Draw a right-angled triangle showing the given information. Label the required height x.



2 The opposite and adjacent sides are involved, so use $\tan\theta$.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

3 Substitute in the known values.

$$\tan 29^\circ = \frac{x}{8}$$

4 Multiply both sides by 8.

$$8 \times \tan 29^\circ = x$$

5 Use your calculator to find the value of x.

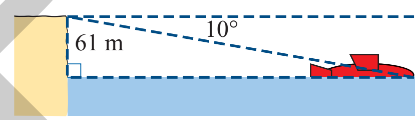
$$x = 4.434\ldots$$

6 Write your answer correct to 1 decimal place.

The height of the ash plume was 4.4 km.

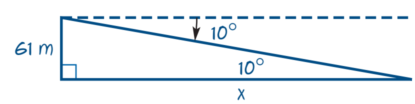
Example 10 Angle of depression

From the top of a cliff 61 m above sea-level, Chen saw a capsized yacht. He estimated the angle of depression to be about 10°. How far was the yacht from the base of the cliff, to the nearest metre?



Solution

1 Draw a diagram showing the given information. Label the required distance x.



2 Mark in the angle at the yacht corner of the triangle. This is also 10°, because it and the angle of depression are alternate (or 'Z') angles.

Warning: The angle between the cliff face and the line of sight is *not* 10°.

3 The opposite and adjacent sides are involved, so use $\tan\theta$.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

4 Substitute in the known values.

$$\tan 10^\circ = \frac{61}{x}$$

5 Multiply both sides by x.

$$x \times \tan 10^\circ = 61$$

6 Divide both sides by $\tan 10^\circ$.

$$x = \frac{61}{\tan 10^\circ}$$

7 Do the division using your calculator.

$$x = 345.948\ldots$$

8 Write your answer to the nearest metre.

The yacht was 346 m from the base of the cliff.

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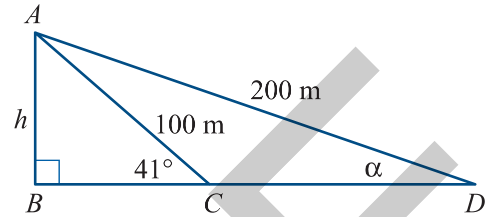
Example 11 Application with two right-angled triangles

A cable 100 m long makes an angle of elevation of 41° with the top of a tower.

a Find the height h of the tower, to the nearest metre.

b Find the angle of elevation α, to the nearest degree, that a cable 200 m long would make with the top of the tower.

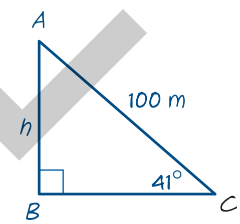


Solution

Strategy: Find h in triangle ABC, then use this value to find α in triangle ABD.

a

1 Draw triangle ABC showing the given and required information.



2 The opposite and hypotenuse are involved, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

3 Substitute in the known values.

$$\sin 41^\circ = \frac{h}{100}$$

4 Multiply both sides by 100.

$$h = 100 \times \sin 41^\circ$$

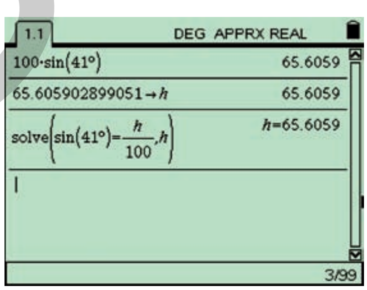
5 Evaluate 100 sin(41°) using your calculator and store the answer as the value of the variable h for later use.

$$h = 65.605.$$

TI-nspire

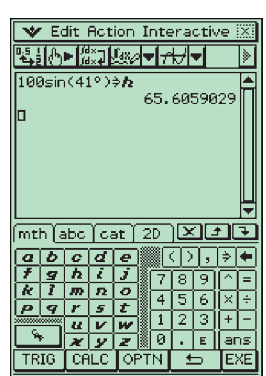
a Type in 1 0 0 × sin 4 1 ctrl ° and press enter to evaluate 100 sin(41°)(= 65.605 . . .).

b Press ctrl sto→ H enter to store 65.605 . . . as the value of the variable h.



ClassPad

From the TRIG menu of the mth tab, tap the following sequence of buttons: 1 0 0 sin 4 1 °) ⇒ VAR and select h. Press EXE to calculate 100 sin(41°)(= 65.605 . . .) and store the answer as the value of the variable h.



6 Write your answer to the nearest metre.

The height of the tower is 66 m.

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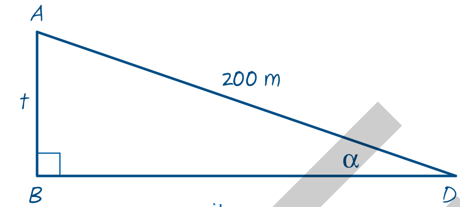
b

1 Draw triangle ABD showing the given and required information



2 The opposite and hypotenuse are involved, so use $\sin\theta$.

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

3 Substitute in the known values. In part **a** we stored the height of the tower as T.

$\sin\alpha = \dfrac{t}{200}$

4 Write the equation to find α.

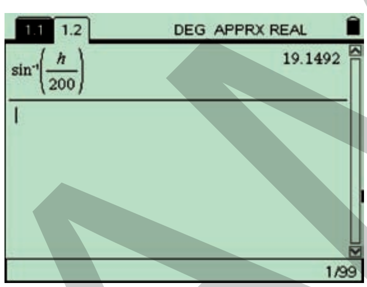
$\alpha = \sin^{-1}\left(\dfrac{t}{200}\right)$

5 Use your calculate to evaluate α.

$\alpha = 19.149\ldots$

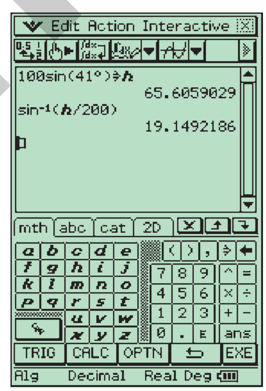
TI-nspire

Type in ctrl sin H ÷ 2 0 0 enter and press enter to find the value of α (= 19.149 . . .).



ClassPad

Tap the following sequence of buttons: sin⁻¹ h ÷ 2 0 0) and press EXE to find the value of α (= 19.149 . . .).

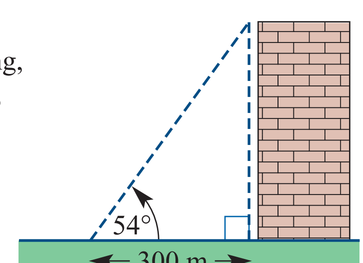


6 Write your answer to the nearest degree.

The 200 m cable would have an angle of elevation of 19°.

Exercise 7E

1 After walking 300 m away from the base of a tall building, on level ground, Elise measured the angle of elevation to the top of the building to be 54°. Find the height of the building, to the nearest metre.



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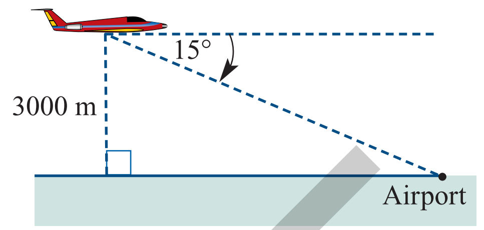
2 The pilot of an aeroplane saw an airport at sea-level at an angle of depression of 15°. His altimeter showed that the aeroplane was at a height of 3000 m. Find the horizontal distance of the aeroplane from the airport, to the nearest metre.



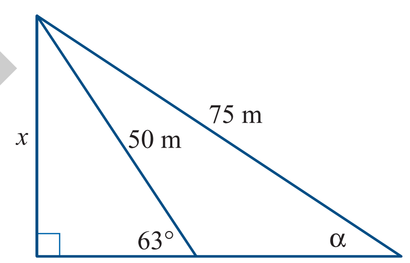
3 The angle of elevation measured from ground level to the top of a tall tree was 41°. The distance of the measurer from the base of the tree was 38 m. How tall was the tree? Give your answer to the nearest metre.

4 When Darcy looked from the top of a cliff, 60 m high, he noticed his girlfriend at an angle of depression of 20° on the ground below. How far was she from the cliff? Answer correct to 1 decimal place.

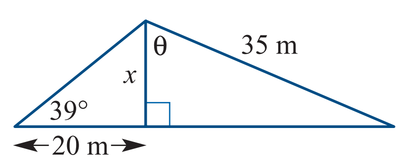
5 From the top of a mountain I could see a town at an angle of depression of 1.4° across the level plain. Looking at my map I found that the town was 10 km away. Find the height of the mountain above the plain, to the nearest metre.

6 What would be the angle of elevation to the top of a radio transmitting tower 100 m tall and 400 m from the observer? Answer to the nearest degree.

7 a Find the length x, correct to 1 decimal place.
b Find the angle α, to the nearest degree.

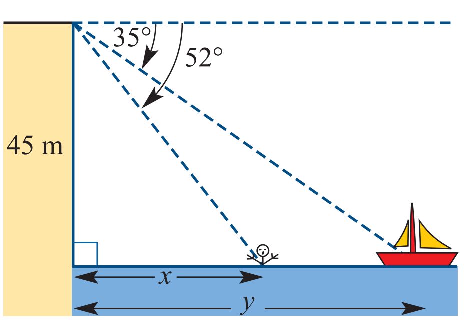


8 a Find the length x, correct to 1 decimal place.
b Find the angle θ, to the nearest degree.



9 From the top of a cliff 45 m high, an observer looking along an angle of depression of 52° could see a man swimming in the sea. The observer could also see a boat at an angle of depression of 35°. Calculate to the nearest metre:

a the distance x of the man from the base of the cliff
b the distance y of the boat from the base of the cliff
c the distance from the man to the boat.



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10 A police helicopter hovering in a fixed position at an altitude of 500 m moved its spotlight through an angle of depression of 57° onto a lost child. The pilot sighted the rescue team at an angle of depression of 31°. If the terrain was level, how far, to the nearest metre, was the rescue team from the child?

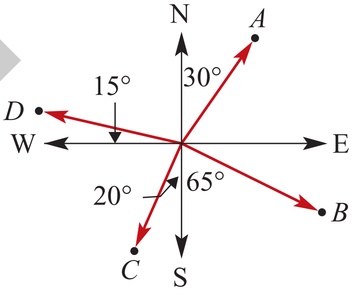
7.6 Bearings and navigation

Compass bearings

A **compass bearing** gives the direction by stating the angle either side of north or south. For example, a compass bearing of N40°E is found by facing north and then swinging 40° towards the east side.

Example 12 **Determining compass bearings**

Give the compass bearings of the points A, B, C and D.

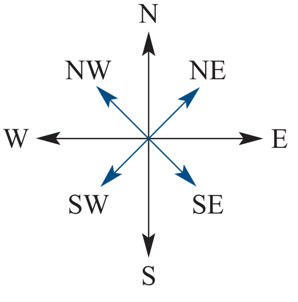


Solution

- To find the direction of A, face north and swing 30° east. — A is in the direction N30°E.
- To find the direction of B, face south and swing 65° east. — B is in the direction S65°E.
- To find the direction of C, face south and swing 20° west. — C is in the direction S20°W.
- To find the direction of D, face north and swing west. — Angle from north $= 90° - 15°$
  $= 75°$
  D is in the direction N75°W.

Directions midway between the four directions of the compass combine the letters of the directions they are between. For example, the direction midway between north and east is often called north-east (NE).

It could also be called N45°E.



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True bearings

A **true bearing** is the angle measured clockwise from north around to the required direction. True bearings are sometimes called **three-figure bearings** because they are written using three numbers or figures. For example, 090°T is the direction measured 90° clockwise from north, better known as east!

Example 13 Determining true bearings from compass bearings

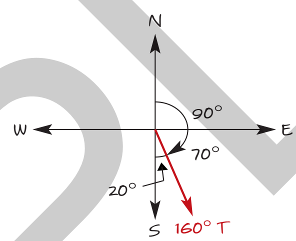
Describe the compass bearings below as true bearings:

a S20°E **b** N80°W

Solution

a

1. Show the direction on the diagram of the compass points.
2. Add the angles clockwise from north to the required direction.
   Note that the four points of a compass are 90° apart.

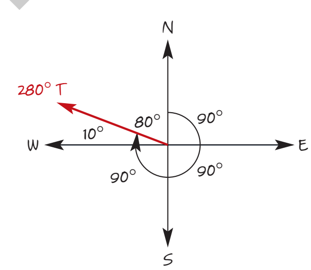


3. Write your answer.

Bearing $= 90° + 70° = 160°\text{T}$
The true bearing is 160°T.

b

1. Show the direction on the diagram of the compass points.
2. Add the angles clockwise from north to the required direction.
   or
   The direction is 80° less than one full sweep (360°) of the compass.



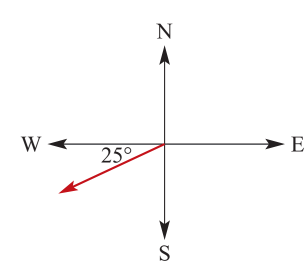
Bearing $= 90° + 90° + 90° + 10° = 280°\text{T}$
or $= 360° - 80° = 280°\text{T}$

3. Write your answer.

The true bearing is 280°T.

Example 14 Determining compass and true bearings

Give the compass bearing and true bearing for the direction shown.



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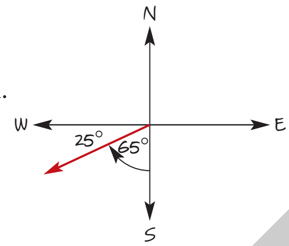
Solution

Compass bearing

1 Calculate the angle from the direction of south. Notice that the swing is towards west.



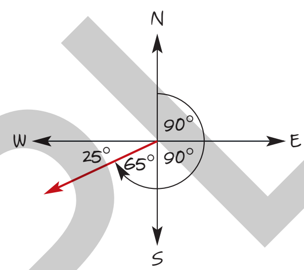
2 Write your answer.

Angle from south $= 90^\circ - 25^\circ = 65^\circ$

The compass bearing is S65°W.

True bearing

1 Calculate the total angles swept out clockwise from north.
There is an angle of 90° between each of the four points of the compass.



True bearing $= 90^\circ + 90^\circ + 65^\circ = 245^\circ$T

or $270^\circ - 25^\circ = 245^\circ$T

2 Write your answer.

The true bearing is 245°T.

Navigation problems

Navigation problems usually involve a consideration of not only the **direction** of travel, given as a bearing, but also the **distance** travelled.

In many practical applications we need to know the **distance** that has been travelled after moving at a particular **speed** for a given **time**. If a car moved at 60 km/h for 2 hours, the distance travelled would be $2 \times 60 = 120$ km.

Distance travelled and speed

When travelling at a constant speed:

$$\text{Distance travelled} = \text{time taken} \times \text{speed}$$

Make sure that the same units of length and time are used for the speed, distance and time. If a car moved at 60 km/h for 90 minutes, convert 90 minutes to 1.5 hours before multiplying by the speed. The distance travelled would be $1.5 \times 60 = 90$ km.

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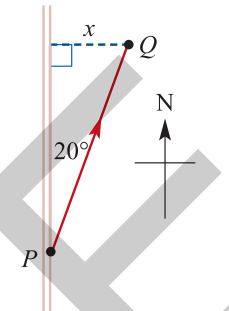
Example 15 Navigating using a compass bearing

A group of bushwalkers leave point P, which is on a road that runs north–south, and walk for 6 hours in the direction N20°E to reach point Q. They walk at 5 km/h.

a What is the shortest distance x from Q back to the road correct to 1 decimal place?

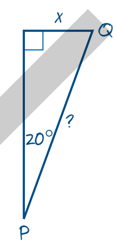
b Looking from point Q, what would be the compass bearing and true bearing of their starting point?



Solution

a

1 Show the given and required information in a right-angled triangle.



2 Calculate the distance travelled, PQ. Distance = time taken × speed.

$$\text{Distance } PQ = 6 \text{ hours} \times 5 \text{ km/h}$$
$$= 30 \text{ km}$$

3 The opposite and hypotenuse are involved, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

4 Substitute in the known values.

$$\sin 20^\circ = \frac{x}{30}$$

5 Multiply both sides by 30.

$$30 \times \sin 20^\circ = x$$

6 Find the value of x using your calculator.

$$x = 10.260\ldots$$

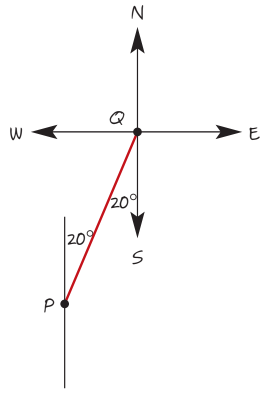
7 Write your answer correct to 1 decimal place.

The shortest distance to the road is 10.3 km.

b

1 Draw the compass points at Q.

2 Enter the alternate angle 20°.



3 The direction of P, looking from Q, is given by a swing of 20° from south towards west.

The compass bearing is S20°W.

4 Standing at Q, add all the angles when facing north and then turning clockwise to look at P. This gives the true bearing of P when looking from Q.

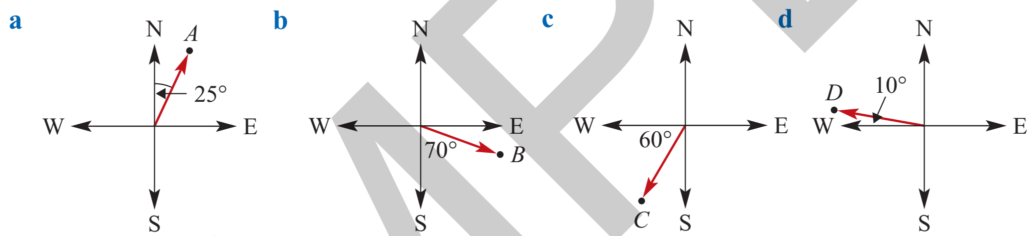
The true bearing is $180° + 20° = 200°\text{T}$.

Exercise 7F

1 Give the compass bearing (from north or south) and the true bearing of each of the directions:

a SE **b** SW **c** NW

2 State the compass bearing and true bearing of each of the points A, B, C and D.

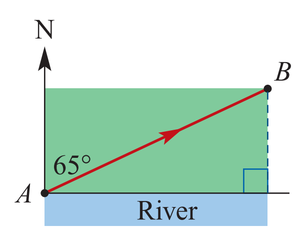


3 Eddie camped overnight at point A beside a river that ran east–west. He walked in the direction N65°E for 3 hours to point B. Eddie walks at 6 km/h.

a What angle did his direction make with the river?

b How far did he walk from A to B?

c What is the shortest distance from B to the river, correct to 2 decimal places?



4 A ship sailed 3 km west, then 2 km south.

a Give its compass bearing from an observer who stayed at its starting point, correct to 1 decimal place.

b For a person on the ship, what would be the compass bearing looking back to the starting point?

5 An aeroplane flew 500 km south, then 600 km east. Give its true bearing from its starting point, to the nearest degree.

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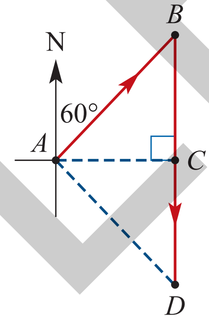
6 A ship left port and sailed east for 5 km, then sailed north. After some time an observer at the port could see the ship in the direction N50°E.

a How far north had the ship travelled? Answer correct to 1 decimal place.
b Looking from the ship, what would be the true bearing of the port?

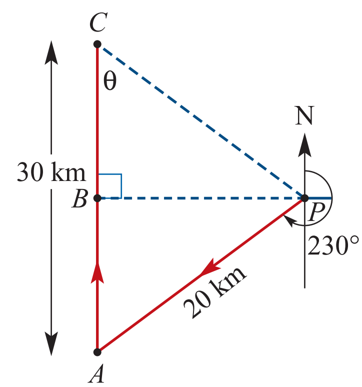
7 A woman walked from point A for 2 hours in the direction N60°E to reach point B. Then she walked for 3 hours heading south until she was at point D. The woman walked at a constant speed of 5 km/h. Give the following distances correct to 1 decimal place and directions to the nearest degree.

a Find the distances walked from A to B and from B to D.
b How far south did she walk from B to C?
c Find the distance from A to C.
d What is the distance from C to D?
e Find the compass bearing and distance she would need to walk to return to her starting point.



8 A ship left port P and sailed 20 km in the direction 230°T. It then sailed north for 30 km to reach point C. Give the following distances correct to 1 decimal place and directions to the nearest degree.

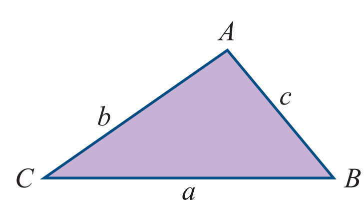
a Find the distance AB.
b Find the distance BP.
c Find the distance BC.
d Find the angle θ at point C.
e State the true bearing and distance of the port P from the ship at C.



7.7 The sine rule

Standard triangle notation

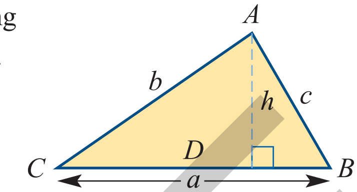
The convention for labelling a non-right-angled triangle is to use the upper case letters A, B, and C for the angles at each corner. The sides are named using lower case letters so that side a is opposite angle A, and so on.



This notation is used for the sine rule and cosine rule (see Section 7.8). Both rules can be used to find angles and sides in triangles that do not have a right angle.

How to derive the sine rule

In triangle ABC, show the height h of the triangle by drawing a perpendicular line from D on the base of the triangle to A.



In triangle ADC, $\sin C = \dfrac{h}{b}$

So $h = b \times \sin C$

In triangle ABD, $\sin B = \dfrac{h}{c}$

So $h = c \times \sin B$

We can make the two rules for h equal to each other. $b \times \sin C = c \times \sin B$

Divide both sides by $\sin C$. $b = \dfrac{c \times \sin B}{\sin C}$

Divide both sides by $\sin B$. $\dfrac{\boldsymbol{b}}{\mathbf{sin}\ \boldsymbol{B}} = \dfrac{\boldsymbol{c}}{\mathbf{sin}\ \boldsymbol{C}}$

If the triangle was redrawn with side c as the base, then using similar steps we would get: $\dfrac{\boldsymbol{a}}{\mathbf{sin}\ \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\mathbf{sin}\ \boldsymbol{B}}$

We can combine the two rules as shown in the following box.

The sine rule

In any triangle ABC;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The sine rule can be used to find the sides and angles in a non-right-angled triangle when given:

- two sides and an angle opposite one of the given sides

or

- two angles and one side*.

*If neither of the given angles is opposite the given side, find the third angle using $A + B + C = 180°$.

The sine rule is really three possible equations:

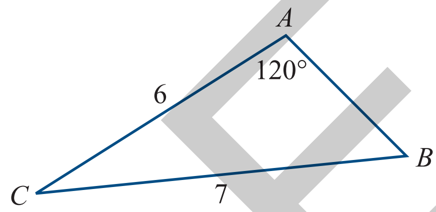
$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

Each equation has two sides and two angles opposite those sides. If we know three of the parts, we can find the fourth. So if we know two angles and a side opposite one of the angles, we can find the side opposite the other angle. Similarly, if we know two sides and an angle opposite one of the sides, we can find the angle opposite the other side.

Using the sine rule

Example 16 **Using the sine rule given two sides and an opposite angle**

Find angle B in the triangle shown, correct to 1 decimal place.



Solution

1 We have the pairs $a = 7$ and $A = 120^\circ$
$b = 6$ and $B = ?$
with only B unknown.
So use $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

2 Substitute in the known values.

$$\frac{7}{\sin 120^\circ} = \frac{6}{\sin B}$$

3 Cross-multiply.

$$7 \times \sin B = 6 \times \sin 120^\circ$$

4 Divide both sides by 7.

$$\sin B = \frac{6 \times \sin 120^\circ}{7}$$

5 Write the equation to find angle B.

$$B = \sin^{-1}\left(\frac{6 \times \sin 120^\circ}{7}\right)$$

6 Use your calculator to evaluate the expression for B.

$$B = 47.928\ldots^\circ$$

7 Write your answer correct to 1 decimal place.

Angle B is 47.9°.

In Example 16, now that we know that $A = 120^\circ$ and $B = 47.9^\circ$, we can use the fact that the angles in a triangle add to 180° to find C.

$$\begin{aligned} A + B + C &= 180^\circ \\ 120^\circ + 47.9^\circ + C &= 180^\circ \\ 167.9^\circ + C &= 180^\circ \\ C &= 180^\circ - 167.9^\circ = 12.1^\circ \end{aligned}$$

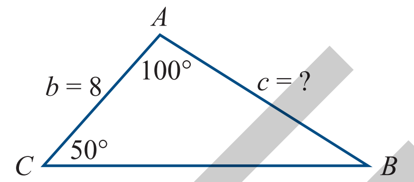
As we now know that $A = 120^\circ$, $a = 7$ and $C = 12.1^\circ$, we can find side c using $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

The steps are similar to those in the example.

Finding all the angles and sides of a triangle is called **solving the triangle**.

Example 17 **Using the sine rule given two angles and one side**

Find side c in the triangle shown, correct to 1 decimal place.



Solution

1 Find the angle opposite the given side by using $A + B + C = 180°$

$$A + B + C = 180°$$
$$100° + B + 50° = 180°$$
$$B + 150° = 180°$$
$$B = 30°$$

2 We have the pairs $b = 8$ and $B = 30°$
$c = ?$ and $C = 50°$

with only c unknown. So use $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

3 Substitute in the known values.

$$\frac{8}{\sin 30°} = \frac{c}{\sin 50°}$$

4 Multiply both sides by sin 50°.

$$c = \frac{8 \times \sin 50°}{\sin 30°}$$

5 Use your calculator to find c.

$$c = 12.256\ldots$$

6 Write your answer correct to 1 decimal place.

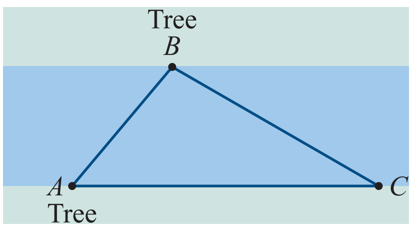
Side c is 12.3 units long

In some special cases it is possible to draw two different triangles that *both* fit the given information. This is called the **ambiguous case** of the sine rule. It is covered in the *Essential Further Mathematics* textbook.

Example 18 **Application of the sine rule**

Leo wants to tie a rope from a tree at point A to a tree at point B on the other side of the river. He needs to know the length of rope required. When he stood at A, the compass bearing of B was N40°E. Leo walked 200 m east along the river bank to C, where the compass bearing of B was N60°W.

Find the length of rope required to reach from A to B, correct to 2 decimal places.



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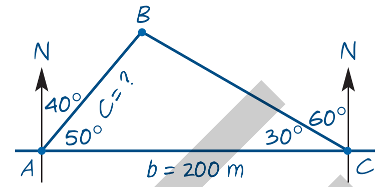
Solution

1 Include the given information in a sketch.



2 Use the compass bearings to find the angle A and the angle C of the triangle.

Angle $A = 90° - 40° = 50°$
Angle $C = 90° - 60° = 30°$

3 To use the sine rule, we need to know an angle and its opposite side. We know side $b = 200$. Use $A + B + C = 180°$ to find angle B.

$$A + B + C = 180°$$
$$50° + B + 30° = 180°$$
$$B = 100°$$

4 We have the pairs: $b = 200$ and $B = 100°$
$c = ?$ and $C = 30°$

with only c unknown. So use $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

5 Substitute in the known values.

$$\frac{c}{\sin 30°} = \frac{200}{\sin 100°}$$

6 Multiply both sides by $\sin 30°$.

$$c = \frac{200 \times \sin 30°}{\sin 100°}$$

7 Use your calculator to find c.

$$c = 101.542\ldots$$

8 Write your answer correct to 2 decimal places.

The rope must be 101.54 m long.

Tips for solving trigonometry problems

- Always make a rough sketch in pencil as you read the details of a problem. You may need to make changes as you read more, but it is very helpful to have a sketch to guide your understanding.
- In any triangle, the longest side is opposite the largest angle. The shortest side is opposite the smallest angle.
- When you have found a solution, re-read the question and check that your answer fits well with the given information and your diagram.

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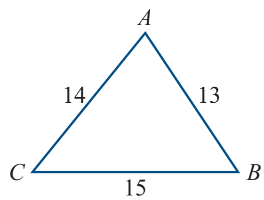
Exercise 7G

In this exercise, calculate lengths correct to 2 decimal places and angles correct to 1 decimal place where necessary.

1 In each triangle, state the lengths of sides a, b and c.

a



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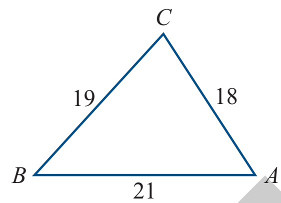
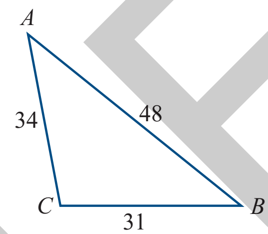
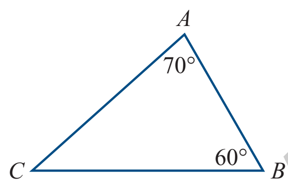


c



2 Find the value of the unknown angle in each triangle. Use $A + B + C = 180°$.

a



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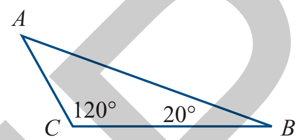
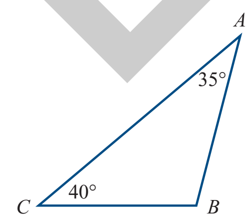


c

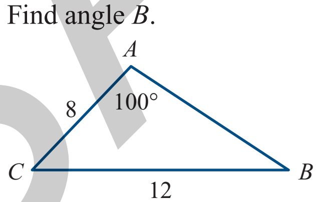


3 In each of the following a student was using the sine rule to find an unknown part of a triangle, but was unable to complete the final steps of the solution. Find the unknown value by completing each problem.

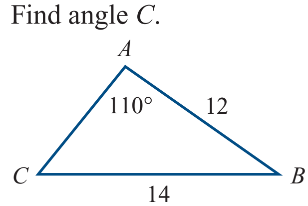
a $\dfrac{a}{\sin 40°} = \dfrac{8}{\sin 60°}$ **b** $\dfrac{b}{\sin 50°} = \dfrac{15}{\sin 72°}$ **c** $\dfrac{c}{\sin 110°} = \dfrac{24}{\sin 30°}$

d $\dfrac{17}{\sin A} = \dfrac{16}{\sin 70°}$ **e** $\dfrac{26}{\sin B} = \dfrac{37}{\sin 95°}$ **f** $\dfrac{21}{\sin C} = \dfrac{47}{\sin 115°}$

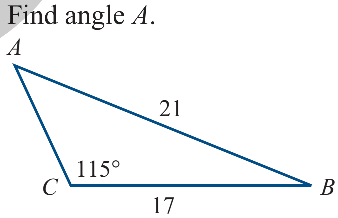
4 **a** Find angle B.



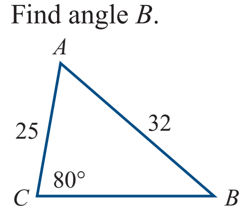
b Find angle C.



c Find angle A.



d Find angle B.



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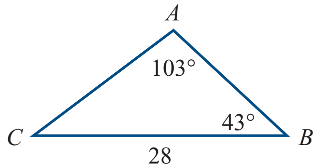
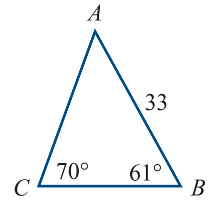
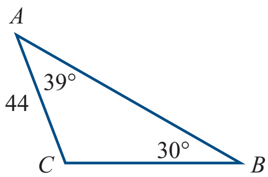
5 **a** Find side b.



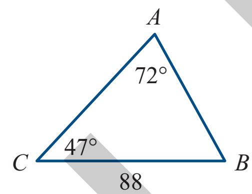
b Find side b.



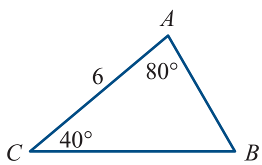
c Find side a.



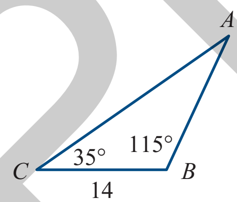
d Find side c.



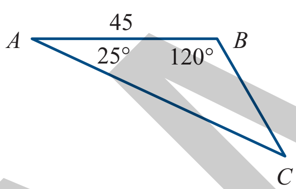
6 **a** Find side c.



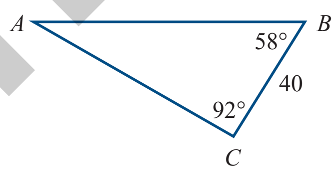
b Find side c.



c Find side b.

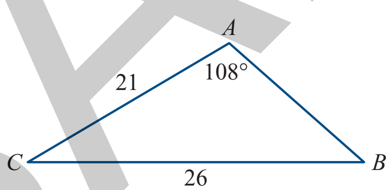


d Find side b.

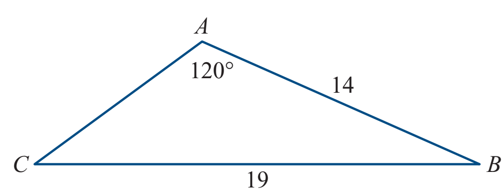


7 Solve (find all the unknown sides and angles of) the following triangles.

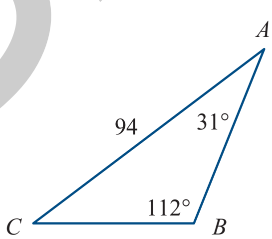
a



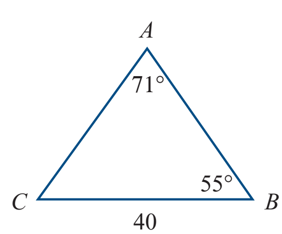
b



c



d



8 In the triangle ABC, $A = 105°$, $B = 39°$ and $a = 60$. Find side b.

9 In the triangle ABC, $A = 112°$, $a = 65$ and $c = 48$. Find angle C.

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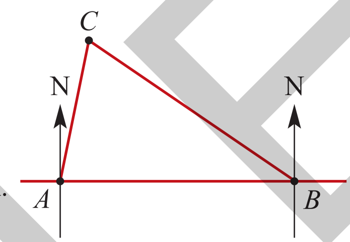
10 In the triangle ABC, $B = 50°$, $C = 45°$ and $a = 70$. Find side c.

11 In the triangle ABC, $B = 59°$, $C = 74°$ and $c = 41$. Find sides a and b and angle A.

12 In the triangle ABC, $a = 60$, $b = 100$ and $B = 130°$. Find angles A and C and side c.

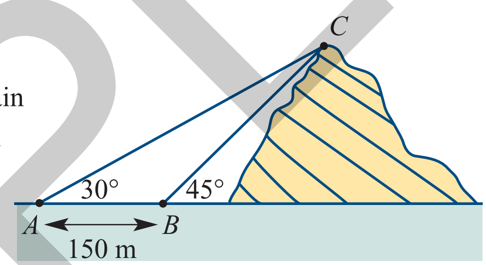
13 In the triangle ABC, $A = 130°$, $B = 30°$ and $c = 69$. Find sides a and b and angle C.

14 A firespotter located in a tower at A saw a fire in the direction N10°E. Five kilometres to the east of A another firespotter at B saw the fire in the direction N60°W.



a Copy the diagram and include the given information.
b Find the distance of the fire from each tower.

15 A surveyor standing at point A measured the angle of elevation to the top of the mountain as 30°. She moved 150 m closer to the mountain and at point B measured the angle of elevation to the top of the mountain as 45°.



There is a proposal to have a strong cable from point A to the top of the mountain to carry tourists in a cable car. What is the length of the required cable?

16 A naval officer sighted the smoke of a volcanic island in the direction N44°E. A navigator on another ship 25 km due east of the first ship saw the smoke in the direction N38°W.

a Find the distance of each ship from the volcano.
b If the ship closest to the volcano can travel at 15 km/h, how long will it take it to reach the volcano?

17 An air-traffic controller at airport A received a distress call from an aeroplane low on fuel. The bearing of the aeroplane from A was 070°T. From airport B, 80 km north of airport A, the bearing of the aeroplane was 120°T.

a Which airport was closest for the aeroplane?
b Find the distance to the closest airport.

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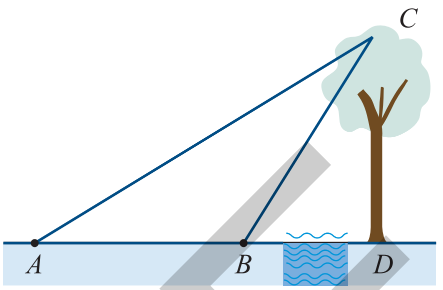
18 Holly was recording the heights of tall trees in a State forest to have them registered for protection. A river prevented her from measuring the distance from the base of a particular tree.

She recorded the angle of elevation of the top of the tree from point A as 25°. Holly walked 80 m towards the tree and recorded the angle of elevation from point B as 50°.



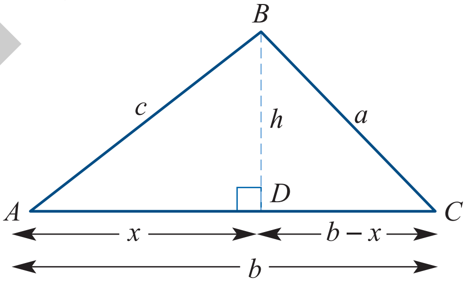
a Copy the diagram shown and add the given information.
b Find the angle at B in triangle ABC.
c Find the angle at C in triangle ABC.
d Find the length b (from A to C).
e Use the length b as the hypotenuse in right-angled triangle ADC, and the angle at A, to find distance DC, the height of the tree.

7.8 The cosine rule

The cosine rule can be used to find the length of a side in any non-right-angled triangle when two sides and the angle between them are known. When you know the three sides of a triangle, the cosine rule can be used to find any angle.

How to derive the cosine rule

In the triangle ABC, show the height h of the triangle by drawing a line perpendicular from D on the base of the triangle to B.



Let $AD = x$
As $AC = b$, then $DC = b - x$.

In triangle ABD, $\cos A = \dfrac{x}{c}$
Multiply both sides by c. $x = c\cos A$ ①
Using Pythagoras' Theorem in triangle ABD. $x^2 + h^2 = c^2$ ②
Using Pythagoras' Theorem in triangle CBD. $(b-x)^2 + h^2 = a^2$
Expand (multiply out) the squared bracket. $b^2 - 2bx + x^2 + h^2 = a^2$
Use ① to replace x with $c\cos A$. $b^2 - 2bc\cos A + x^2 + h^2 = a^2$
Use ② to replace $x^2 + h^2$ with c^2. $b^2 - 2bc\cos A + c^2 = a^2$
Reverse and rearrange the equation. $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$
Repeating these steps with side c as the base, we get: $\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$
Repeating these steps with side a as the base, we get: $\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$

The three versions of the cosine rule can be rearranged to give rules for $\cos A$, $\cos B$, and $\cos C$.

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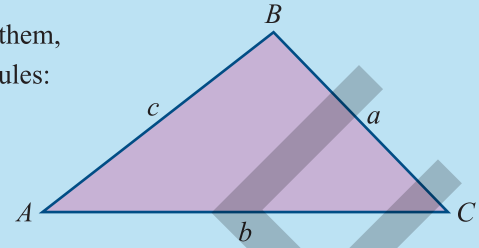
The cosine rule

In any triangle ABC:

- when given two sides and the angle between them, the third side can be found using one of the rules:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

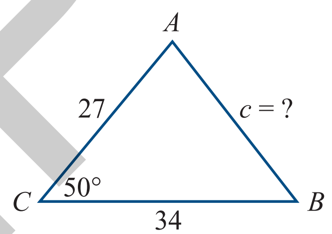


- when given three sides, any angle can be found using one of the following rearrangements of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 19 **Using the cosine rule given two sides and the angle between them**

Find side c, correct to 2 decimal places, in the triangle shown.



Solution

1 Write down the given values and the required unknown value. — $a = 34$, $b = 27$, $c = ?$, $C = 50°$

2 We are given two sides and the angle between them. To find side c use
$$c^2 = a^2 + b^2 - 2ab\cos C$$
— $c^2 = a^2 + b^2 - 2ab\cos C$

3 Substitute the given values into the rule. — $c^2 = 34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°$

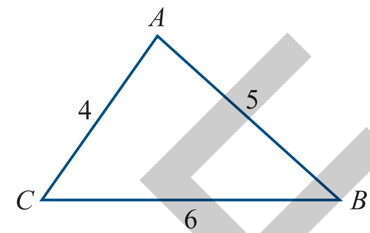
4 Take the square root of both sides. — $c = \sqrt{(34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°)}$

5 Use your calculator to find c. — $c = 26.548\ldots$

6 Write your answer correct to 2 decimal places. — The length of side c is 26.55 units.

Example 20 Using the cosine rule to find an angle given three sides

Find the largest angle, correct to 1 decimal place, in the triangle shown.



Solution

1 Write down the given values.

$a = 6, b = 4, c = 5$

2 The largest angle is always opposite the largest side, so find angle A.

$A = ?$

3 We are given three sides. To find angle A use

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

4 Substitute the given values into the rule.

$$\cos A = \frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}$$

5 Write the equation to find angle A.

$$A = \cos^{-1}\left(\frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}\right)$$

6 Use your calculator to evaluate the expression for A. Make sure that your calculator is in **DEGREE** mode.

Tip: Wrap all the terms in the numerator (top) within brackets. Also put brackets around all of the terms in the denominator (bottom).

$A = 82.819\ldots^\circ$

7 Write your answer.

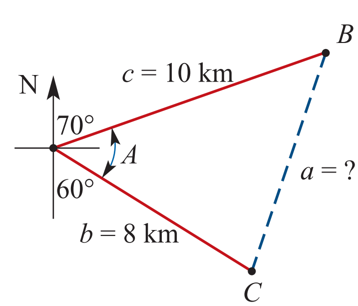
The largest angle is 82.8°.

Example 21 Application of the cosine rule

A bushwalker left his base camp and walked 10 km in the direction N70°E.
His friend also left the base camp but walked 8 km in the direction S60°E.

a Find the angle between their paths.

b How far apart were they when they stopped walking? Give your answer correct to 2 decimal places.



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Solution

a

1 Angles lying on a straight line add to 180°.

$$60^\circ + A + 70^\circ = 180^\circ$$
$$A + 130^\circ = 180^\circ$$
$$A = 50^\circ$$

2 Write your answer.

The angle between their paths was 50°.

b

1 Write down the known values and the required unknown value.

$a = ?,\ b = 8,\ c = 10,\ A = 50^\circ$

2 We have two sides and the angle between them. To find side a use $a^2 = b^2 + c^2 - 2bc\cos A$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

3 Substitute in the known values.

$$a^2 = 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ$$

4 Take the square root of both sides.

$$a^2 = \sqrt{(8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ)}$$

5 Use your calculator to find the value of a.

$$a = 7.820\ldots$$

6 Write your answer correct to 2 decimal places.

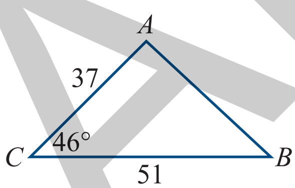
The distance between them was 7.82 km.

Exercise 7H

In this exercise, calculate lengths correct to 2 decimal places and angles correct to 1 decimal place.

1 Find the unknown side in each triangle.

a



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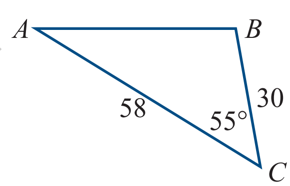

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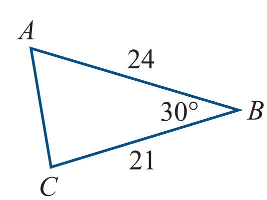
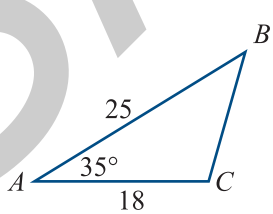
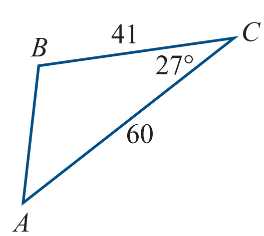


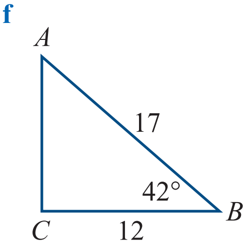
d



e

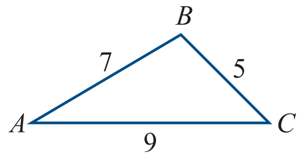


f

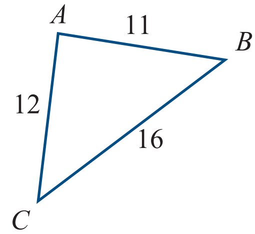


2 Find angle A in each triangle.

a



b



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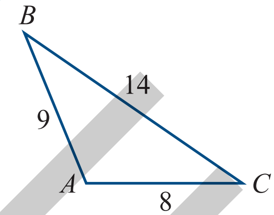
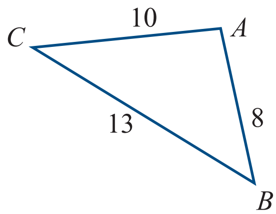


d



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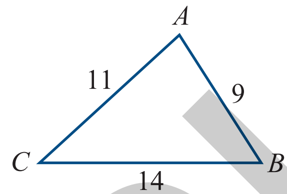

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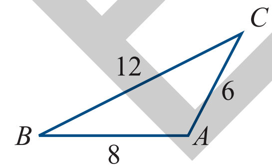


3 In the triangle ABC, $a = 27$, $b = 22$ and $C = 40°$. Find side c.

4 In the triangle ABC, $a = 18$, $c = 15$ and $B = 110°$. Find side b.

5 In the triangle ABC, $b = 42$, $c = 38$ and $A = 80°$. Find side a.

6 In the triangle ABC, $a = 9$, $b = 10$ and $c = 11$. Find angle A.

7 In the triangle ABC, $a = 31$, $b = 47$ and $c = 52$. Find angle B.

8 In the triangle ABC, $a = 66$, $b = 29$ and $c = 48$. Find angle C.

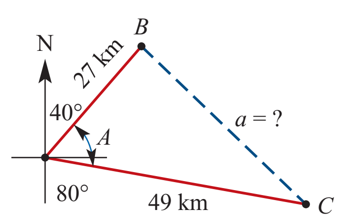
9 Find the smallest angle in the triangle ABC, with $a = 120$, $b = 90$ and $c = 105$.

10 In the triangle ABC, $a = 16$, $b = 21$ and $c = 19$. Find the largest angle.

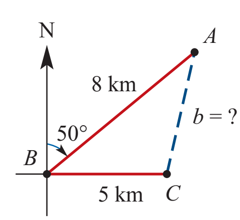
11 A ship left port A and travelled 27 km in the direction N40°E to reach point B.
Another ship left the same port and travelled 49 km in the direction S80°E to arrive at point C.

a Find the angle between the directions of the two ships.

b How far apart were the two ships when they stopped?



12 A battleship B detected a submarine A on a bearing of 050°T and at a distance of 8 km. A cargo ship C was 5 km due east of the battleship.
How far was the submarine from the cargo ship?



13 A farm has a triangular shape with fences of 5 km, 7 km and 9 km in length. Find the size of the smallest angle between the fences.

14 From a lookout tower A, a fire-spotter saw a bushfire B at a distance of 15 km in the direction N45°W. A township C was located 12 km in the direction S85°W. How far was the bushfire from the township?

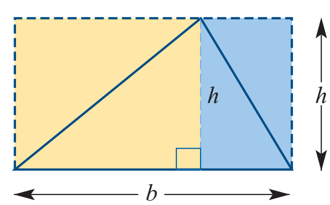
15 Passengers in a car travelling west, along a road that runs east–west, see a mountain 9 km away in the direction N70°W. When they have travelled a further 5 km west along the road, what will be the distance to the mountain?

16 At a point A on the ground, the angle of elevation to the top of a radio transmission tower is 60°. From that point a 40 m cable was attached to the top of the tower. At a point B, a further 10 m away from the base of the tower, another cable is to be pegged to the ground and attached to the top of the tower. What length is required for the second cable?

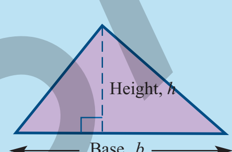
7.9 The area of a triangle

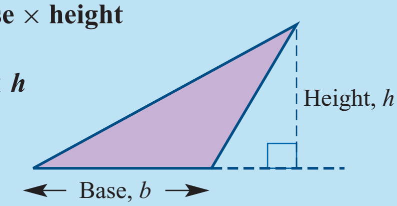
Area of a triangle $= \frac{1}{2}$base $\times$ height

From the diagram, we see that the area of a triangle with a base b and height h is equal to half the area of the rectangle $b \times h$ that it fits within.



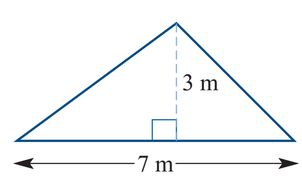
$$\textbf{Area of a triangle} = \frac{1}{2} \times \textbf{base} \times \textbf{height}$$
$$= \frac{1}{2} \times b \times h$$





Example 22 **Finding the area of a triangle using $\frac{1}{2}$ base $\times$ height**

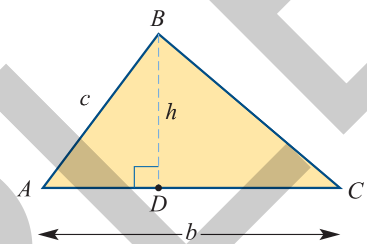
Find the area of the triangle shown, correct to 1 decimal place.



Solution

1 As we are given values for the base and height of the triangle, use Area $= \frac{1}{2} \times$ base $\times$ height	Base, $b = 7$ Height, $h = 3$ Area of triangle $= \frac{1}{2} \times b \times h$
2 Substitute the given values.	$= \frac{1}{2} \times 7 \times 3$
3 Evaluate.	$= 10.5\ \text{m}^2$
4 Write your answer.	The area of the triangle is $10.5\ \text{m}^2$.

Area of a triangle $= \frac{1}{2}bc \sin A$



In triangle ABD, $\sin A = \dfrac{h}{c}$

$$h = c \times \sin A$$

So we can replace h with $c \times \sin A$ in the rule:

$$\text{Area of a triangle} = \tfrac{1}{2} \times b \times h$$
$$\text{Area of a triangle} = \tfrac{1}{2} \times b \times c \times \sin A$$

Similarly, using side c or a for the base, we can make a complete a set of three rules:

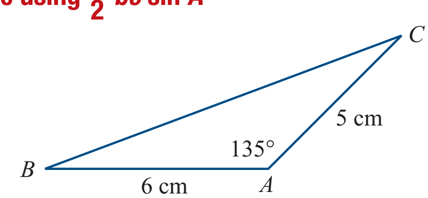
$$\textbf{Area of a triangle} = \tfrac{1}{2}\, bc \sin A$$
$$\textbf{Area of a triangle} = \tfrac{1}{2}\, ac \sin B$$
$$\textbf{Area of a triangle} = \tfrac{1}{2}\, ab \sin C$$

Notice that each version of the rule follows the pattern:

Area of a triangle $= \frac{1}{2} \times$ (*product of two sides*) $\times$ sin(*angle between those two sides*)

Example 23 **Finding the area of a triangle using $\frac{1}{2}bc \sin A$**

Find the area of the triangle shown, correct to 1 decimal place.



Solution

1 We are given two sides b, c and the angle A between them, so use: Area of a triangle $= \frac{1}{2}bc \sin A$	$b = 5,\quad c = 6,\quad A = 135°$ Area of triangle $= \frac{1}{2}bc \sin A$

2 Substitute values for b, c and A into the rule.	$= \frac{1}{2} \times 5 \times 6 \times \sin 135°$
3 Use your calculator to find the area.	$= 10.606\ldots$
4 Write your answer correct to 1 decimal place.	The area of the triangle is 10.6 cm^2.

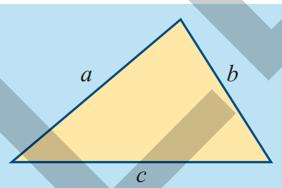
Heron's rule for the area of a triangle

Heron's rule can be used to find the area of any triangle when we know the lengths of the three sides.

Heron's rule for the area of a triangle

Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

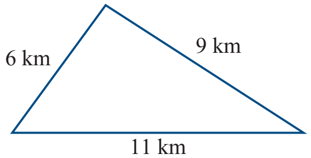
where $s = \frac{1}{2}(a+b+c)$



The variable s is called the **semi-perimeter** because it is equal to half the sum of the sides.

Example 24 **Finding the area of a triangle using Heron's formula**

The boundary fences of a farm are shown in the diagram.
Find the area of the farm, to the nearest square kilometre.



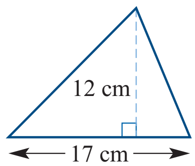
Solution

1 As we are given the three sides of the triangle, use Heron's rule. Start by finding s, the semi-perimeter.	Let $a = 6$, $b = 9$, $c = 11$ $s = \frac{1}{2}(a+b+c)$ $= \frac{1}{2}(6+9+11) = 13$
2 Write Heron's rule.	Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$
3 Substitute the values of s, a, b and c into Heron's rule.	$= \sqrt{13(13-6)(13-9)(13-11)}$
4 Use your calculator to find the area.	$= \sqrt{13 \times 7 \times 4 \times 2}$ $= 26.981\ldots\text{ km}^2$
5 Write your answer.	The area of the farm, to the nearest square kilometre, is 27 km^2.

Exercise 7I

In this exercise, calculate areas correct to 1 decimal place where necessary.

1 Find the area of each triangle.

a


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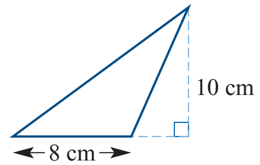

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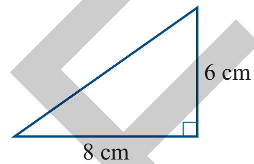

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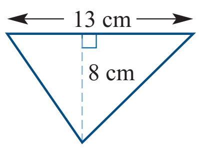

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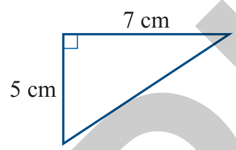

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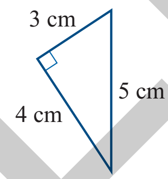
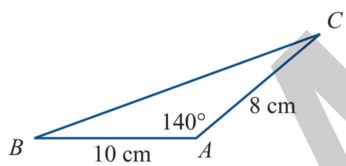


2 Find the areas of the triangles shown.

a


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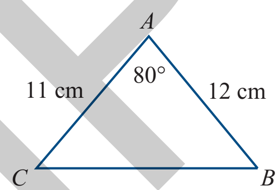

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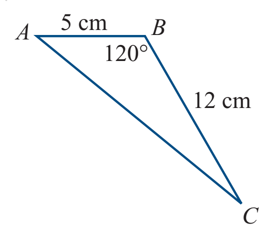

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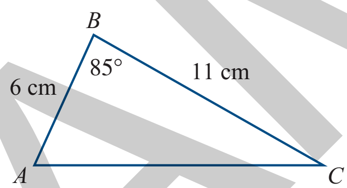

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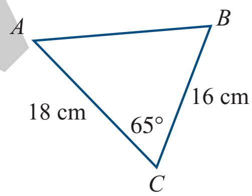

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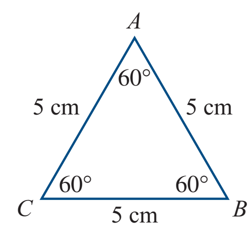
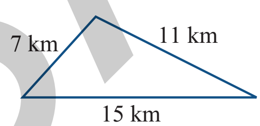


3 Find the area of each triangle.

a


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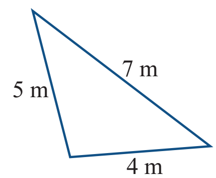

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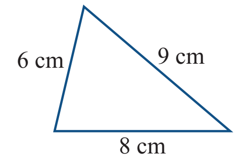

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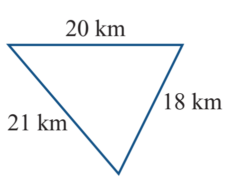

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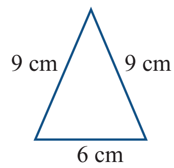

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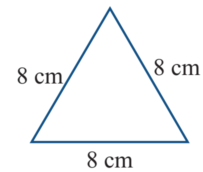

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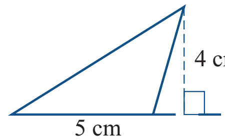
4 Find the area of each triangle shown.

a



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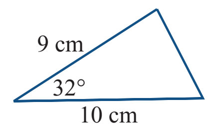

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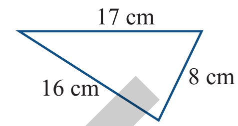

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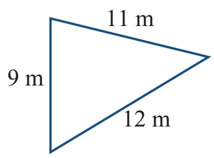

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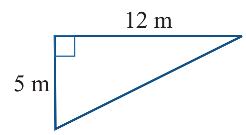

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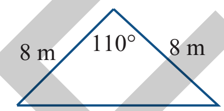

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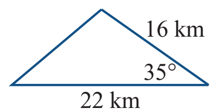

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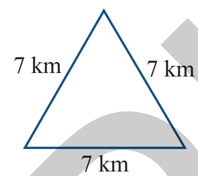

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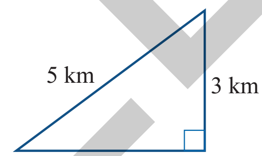
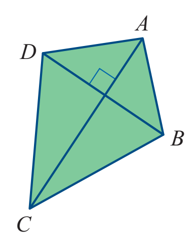


5 Find the area of a triangle with a base of 28 cm and a height of 16 cm.

6 In triangle ABC, side a is 42 cm, side b is 57 cm and angle C is 70°. Find the area of the triangle.

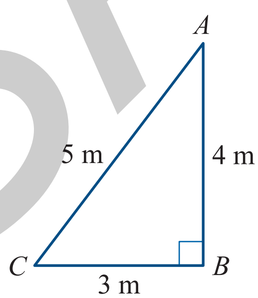
7 Find the area of a triangle with sides of 16 km, 19 km and 23 km.

8 The kite shown is made using two sticks, AC and DB.
The length of AC is 100 cm and the length of DB is 70 cm.
Find the area of the kite.



9 Three students A, B and C stretched a rope loop 12 m long into different shapes. Find the area of each shape.

a



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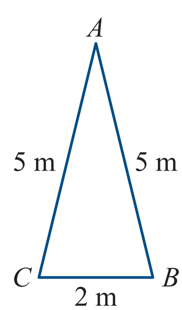

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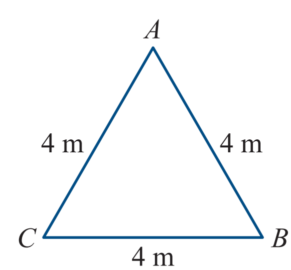


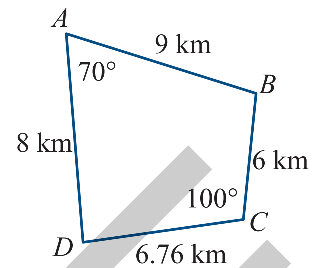
10 A farmer needs to know the area of his property with the boundary fences as shown. The measurements are correct to 2 decimal places.

Hint: Draw a line from *B* to *D* to divide the property into two triangles.

a Find the area of triangle *ABD*.

b Find the area of triangle *BCD*.

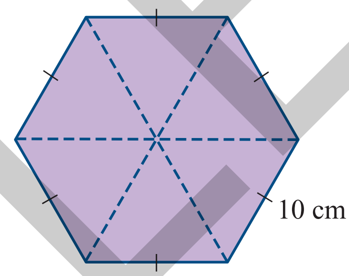
c State the total area of the property.



11 A regular hexagon with sides 10 cm long can be divided into six smaller equilateral triangles. (Remember, an equilateral triangle has all sides of equal length.)

a Find the area of each triangle.

b What is the area of the hexagon?



MC TEST D&D TEST

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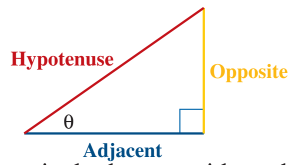
Key ideas and chapter summary

Right-angled triangles

Naming the sides of a right-angled triangle



The **hypotenuse** is the longest side and is always opposite the right angle (90°). The **opposite** side is directly opposite the angle θ (the angle being considered). The **adjacent** side is beside angle θ and runs from θ to the right angle.

Trigonometric ratios

The trigonometric ratios are $\sin\theta$, $\cos\theta$ and $\tan\theta$:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Finding an unknown side in the denominator of the trigonometric ratio

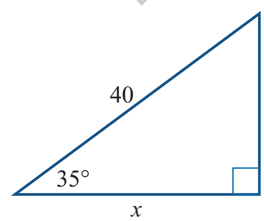
Use the trigonometric ratio that has the given side and the unknown side. Finding x:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 35^\circ = \frac{x}{40}$$

$$x = 40 \times \cos 35^\circ$$

$$x = 32.77$$



Finding an unknown side in the denominator of the trigonometric ratio

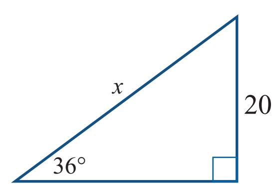
Use the trigonometric ratio that has the given side and the unknown side. Finding x:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 36^\circ = \frac{20}{x}$$

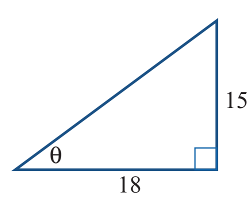
$$x \times \sin 36^\circ = 20$$

$$x = \frac{20}{\sin 36^\circ} = 34.03$$



Finding an unknown angle in a right-angled triangle

Use the trigonometric ratio that has both known sides. After working out the value of the ratio, use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ on your calculator to find the angle.



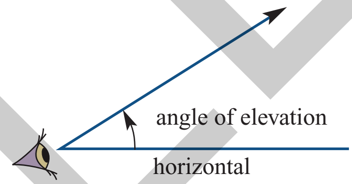
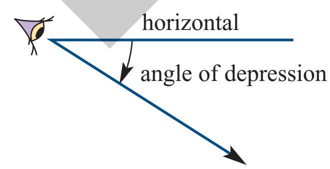
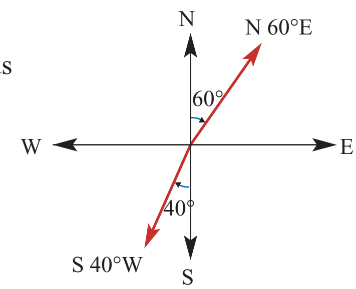
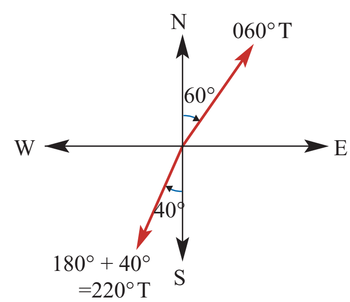
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{15}{18}$$

Review

$\tan\theta = 0.8333$

$\theta = \tan^{-1}(0.8333) = 39.8^\circ$

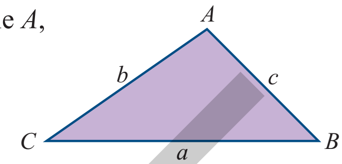
SOH – CAH – TOA	This helps you to remember the trigonometric ratio rules.
Degree mode	Make sure your calculator is in **DEGREE** mode when doing calculations with trigonometric ratios.
Applications of right-angled triangles	Always draw well-labelled diagrams showing all known sides and angles. Also label any sides or angles that need to be found.
Angle of elevation	The angle of elevation is the angle through which you *raise* your line of sight from the horizontal, looking *up* at something. 
Angle of depression	The angle of depression is the angle through which you *lower* your line of sight from the horizontal, looking *down* at something. 
Angle of elevation = angle of depression	The angles of elevation and depression are alternate (‘Z’) angles so are equal.
Compass bearings	Compass bearings are measured by the swing towards west or east from north or south, e.g. N60°E, S40°W. 
True bearings	True bearings are measured clockwise from north and always given with three digits, e.g. 060°T, 220°T. 
Distance, speed and time	Navigation problems usually involve distance, speed and time, as well as direction. Distance travelled = time taken × speed

Review

Non-right-angled triangles

Labelling a non-right-angled triangle

Side a is always opposite angle A, and so on.



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

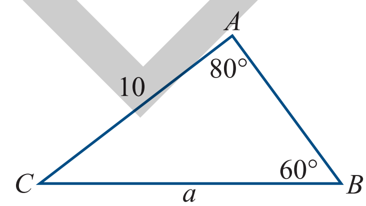
Use the sine rule when given:

- two sides and an angle opposite one of those sides
- two angles and one side.

If neither angle is opposite the given side, find the third angle using $A + B + C = 180^\circ$.

Finding side a correct to one decimal place:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 80^\circ} = \frac{10}{\sin 60^\circ}$$
$$\frac{a}{\sin 80^\circ} = 11.547$$
$$a = 11.547 \times \sin 80^\circ$$
$$= 11.372$$
$$= 11.4$$



Cosine rule

The cosine rule has three versions. When given two sides and the angle between them, use the rule that starts with the required side:

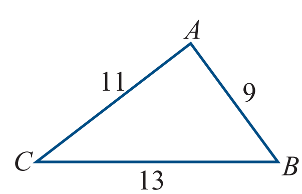
$$a^2 = b^2 + c^2 - 2bc\ \cos A$$
$$b^2 = a^2 + c^2 - 2ac\ \cos B$$
$$c^2 = a^2 + b^2 - 2ab\ \cos C$$

To find an angle when given the three sides, use one of:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Finding angle A.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos A = \frac{11^2 + 9^2 - 13^2}{2 \times 11 \times 9}$$
$$\cos A = 0.1666$$
$$A = \cos^{-1}(0.1666) = 80.4^\circ$$



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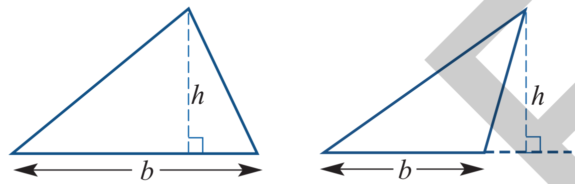
Review

Area of a triangle

Area of △ = $\frac{1}{2}$ × base × height

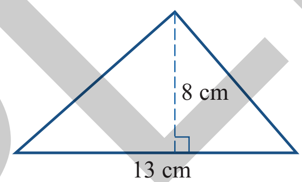
Use this formula if the base and height of the triangle are known:

$$\text{Area of a triangle} = \tfrac{1}{2} \times b \times h$$



Finding the area.

$$\begin{aligned}\text{Area of } \triangle &= \tfrac{1}{2} \times \text{b} \times \text{h}\\ &= \tfrac{1}{2} \times 13 \times 8\\ &= 52 \text{ cm}^2\end{aligned}$$



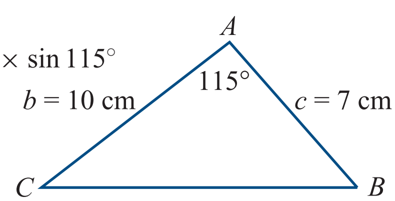
Area of △ = $\frac{1}{2}bc \sin A$

Use this formula if two sides and the angle between them are known. There are three versions of the formula:

$$\text{Area of a triangle} = \tfrac{1}{2}bc\ \sin A$$
$$\text{Area of a triangle} = \tfrac{1}{2}ac\ \sin B$$
$$\text{Area of a triangle} = \tfrac{1}{2}ab\ \sin C$$

Finding the area:

$$\begin{aligned}\text{Area of a } \triangle &= \tfrac{1}{2}\ bc\ \sin A\\ &= \tfrac{1}{2} \times 10 \times 7 \times \sin 115^\circ\\ &= 31.72 \text{ cm}^2\end{aligned}$$



Heron's rule

Use this formula if the lengths of the three sides of the triangle are known:

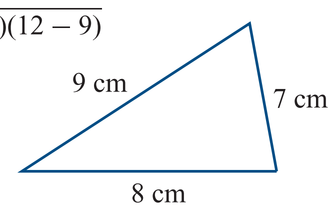
$$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ and is called the **semi-perimeter**.

Finding the area:

$$s = \tfrac{1}{2}(7 + 8 + 9) = 12$$

$$\begin{aligned}\text{Area } \triangle &= \sqrt{12(12-7)(12-8)(12-9)}\\ &= \sqrt{12 \times 5 \times 4 \times 3}\\ &= 26.83 \text{ cm}^2\end{aligned}$$



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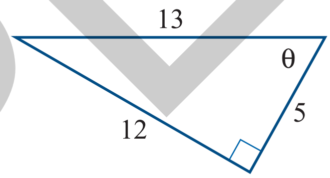
Skills check

Having completed this chapter you should be able to:

- use trigonometric ratios to find an unknown side or angle in a right-angled triangle
- show the angle of elevation or angle of depression on a well-labelled diagram
- show directions on a diagram by using compass bearings or true bearings
- use the sine rule and cosine rule in non-right-angled triangles to find an unknown side or angle
- use the appropriate rule from the three rules for finding the area of a triangle
- solve practical problems involving right-angled and non-right-angled triangles.

Multiple-choice questions

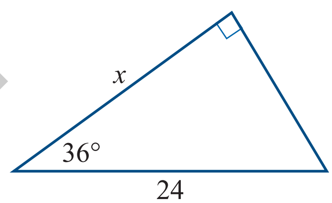
1 In the triangle shown, $\sin\theta$ equals:



A $\dfrac{5}{12}$ **B** $\dfrac{5}{13}$ **C** $\dfrac{13}{12}$

D $\dfrac{12}{13}$ **E** $\dfrac{12}{5}$

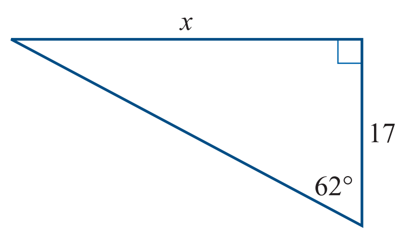
2 The length x is given by:



A $24\sin 36^\circ$ **B** $24\tan 36^\circ$ **C** $24\cos 36^\circ$

D $\dfrac{\sin 36^\circ}{24}$ **E** $\dfrac{\cos 36^\circ}{24}$

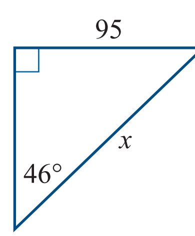
3 To find length x we should use:



A $17\sin 62^\circ$ **B** $17\tan 62^\circ$ **C** $17\cos 62^\circ$

D $\dfrac{\tan 62^\circ}{17}$ **E** $\dfrac{\sin 62^\circ}{17}$

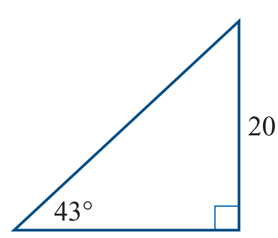
4 The side x is given by:



A $95\tan 46^\circ$ **B** $\dfrac{95}{\cos 46^\circ}$ **C** $\dfrac{\sin 46^\circ}{96}$

D $95\sin 46^\circ$ **E** $\dfrac{95}{\sin 46^\circ}$

5 To find the side x we need to calculate:



A $\dfrac{\tan 43^\circ}{20}$ **B** $\dfrac{20}{\tan 43^\circ}$ **C** $20\tan 43^\circ$

D $20\cos 43^\circ$ **E** $20\sin 43^\circ$

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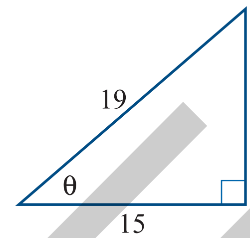
6 To find the angle θ we need to use:

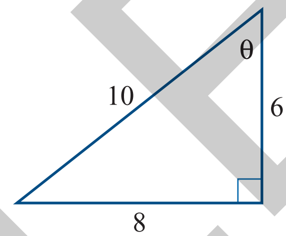
A $\cos^{-1}\left(\frac{15}{19}\right)$ **B** $\cos\left(\frac{15}{19}\right)$ **C** $\sin^{-1}\left(\frac{15}{19}\right)$

D $15 \sin(19)$ **E** $19 \cos(15)$



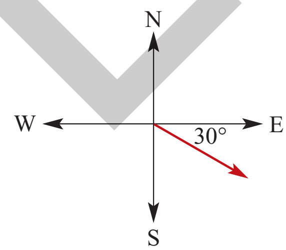
7 The angle θ, correct to 1 decimal place, is:

A 53.1° **B** 36.9° **C** 51.3°
D 38.7° **E** 53.3°



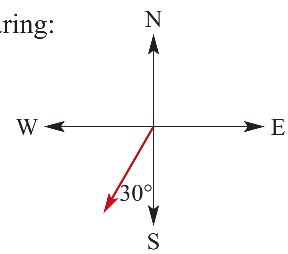
8 The direction shown has the compass bearing:

A N30°S **B** S30°E **C** S60°W
D S60°E **E** N30°E



9 The direction shown could be described as the true bearing:

A 030°T **B** 060°T **C** 210°T
D 150°T **E** −030°T



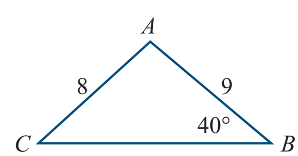
10 A car that travelled for 3 hours at a speed of 60 km/h would cover a distance of:
A 20 km **B** 180 km **C** 63 km **D** 90 km **E** 60 km

11 To find angle C we should use the rule:

A $\frac{a}{\sin A} = \frac{c}{\sin C}$ **B** $\frac{b}{\sin B} = \frac{c}{\sin C}$

C $\frac{a}{\sin A} = \frac{b}{\sin B}$ **D** $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

E $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

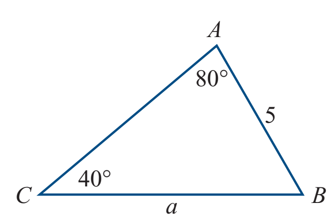


12 To find side a we should use the rule:

A $a^2 = b^2 + c^2$ **B** $a^2 = b^2 + c^2 - 2bc \cos A$

C $\frac{a}{\sin A} = \frac{b}{\sin B}$ **D** $\frac{a}{\sin A} = \frac{c}{\sin C}$

E $\frac{b}{\sin B} = \frac{c}{\sin C}$

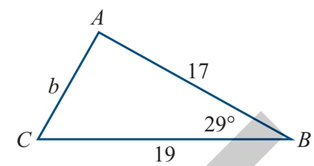


13 The rule needed to find side b is:

A $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ **B** $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

C $c^2 = a^2 + b^2 - 2bc \cos C$ **D** $a^2 = b^2 + c^2$

E $b^2 = a^2 + c^2 - 2ac \cos B$

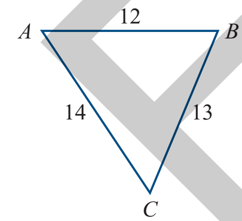


14 To find angle C we should use the rule:

A $\cos C = \dfrac{adjacent}{hypotenuse}$ **B** $\sin C = \dfrac{opposite}{hypotenuse}$

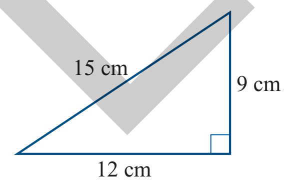
C $\cos C = \dfrac{a^2 + c^2 - b^2}{2ac}$ **D** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

E $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



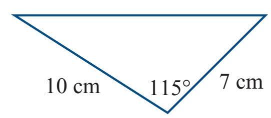
15 The area of the triangle shown is:

A 108 cm^2 **B** 54 cm^2 **C** 36 cm^2
D 90 cm^2 **E** 67.5 cm^2



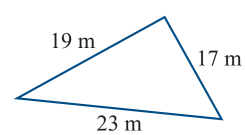
16 The area of the triangle shown, correct to 2 decimal places, is:

A 35.00 cm^2 **B** 70.00 cm^2 **C** 14.79 cm^2
D 31.72 cm^2 **E** 33.09 cm^2



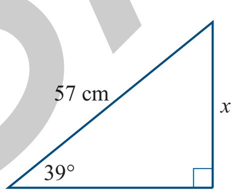
17 The area of the triangle shown, correct to 1 decimal place, is:

A 29.5 m^2 **B** 218.5 m^2 **C** 195.5 m^2
D 161.5 m^2 **E** 158.6 m^2

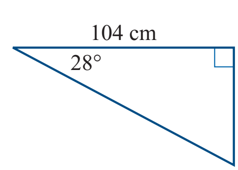


Short-answer questions

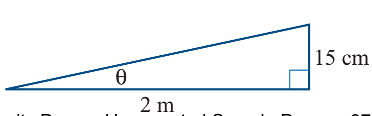
1 Find the length of x, correct to 2 decimal places.



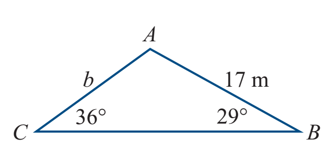
2 Find the length of the hypotenuse, correct to 2 decimal places.



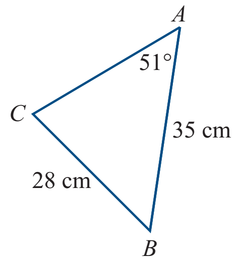
3 A road rises 15 cm for every 2 m travelled horizontally. Find the angle of slope θ, to the nearest degree.



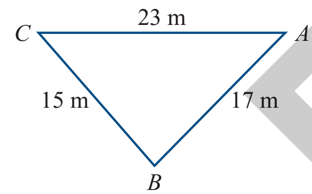
4 Find the length of side b, correct to 2 decimal places.



5 Find the angle C, correct to 1 decimal place.

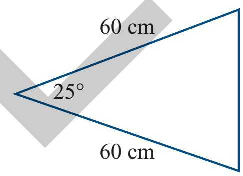


6 Find the smallest angle in the triangle shown, correct to 1 decimal place.



7 A car travelled 30 km east, then travelled 25 km in the direction N70°E. How far was the car from its starting point? Answer correct to 2 decimal places.

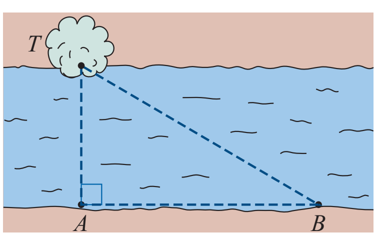
8 A pennant flag is to have the dimensions shown. What area of cloth will be needed for the flag? Answer correct to 1 decimal place.



9 Find the area of an equilateral triangle with sides of 8 m, correct to 1 decimal place.

Extended-response questions

1 Tim was standing at point A when he saw a tree T directly opposite him on the far bank of the river. He walked 100 m along the river bank to point B and noticed that his line of sight to the tree made an angle of 27° with the river bank.



Answer the following correct to 2 decimal places.

a How wide was the river?

b What is the distance from point B to the tree?

Standing at B, Tim measured the angle of elevation to the top of the tree to be 18°.

c Make a clearly labelled diagram showing distance TB, the height of the tree and the angle of elevation, then find the height of the tree.

2 One group of bushwalkers left a road running north–south to walk along a bearing of 060°T. A second group of walkers left the road from a point 3 km further north. They walked on a bearing of 110°T. The two groups met at the point C, where their paths intersected.

a Find the angle at which their paths met.

b Find the distance walked by each group, correct to 2 decimal places.

c If the bushwalkers decided to return to the road by walking back along the path that the second group of walkers had taken, what compass bearing should they follow?

3 A yacht P left port and sailed in the direction N70°W at 15 km/h for 3 hours. Another yacht Q left the same port but sailed in the direction N40°E at 18 km/h for 3 hours.

a How far did yacht P sail?

b How far did yacht Q sail?

c What was the angle between their directions?

d How far apart were they after 3 hours (correct to 2 decimal places)?

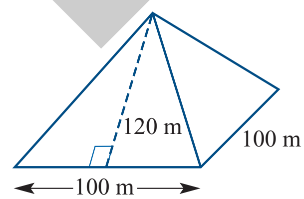
4 A triangular shadecloth must have sides of 5 m, 6 m and 7 m to cover the required area of a children's playground.

a What angle is required in each of the corners (correct to 1 decimal place)?

b The manufacturer charges according to the area of the shadecloth. What is the area of this shadecloth (correct to 2 decimal places)?

c The cost of shadecloth is $29 per square metre. What will be the cost of this shadecloth?

5 The pyramid shown has a square base with sides of 100 m. The line down the middle of each side is 120 m long.

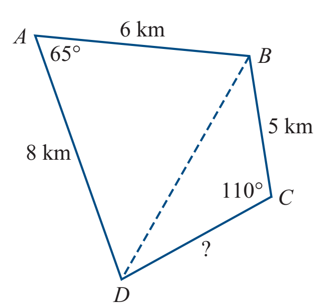


a Find the total surface area of the pyramid. (As the pyramid rests on the ground, the area of its base is not part of its surface area.)

b If 1 kg of gold can be rolled flat to cover 0.5 m^2 of surface area, how much gold would be needed to cover the surface of the pyramid?

c At today's prices, 1 kg of gold costs $15 500. How much would it cost to cover the pyramid with gold?

6 A surveyor measured the boundaries of a property as shown in the diagram. The side CD could not be measured because it crossed through a swamp. The owner of the property wanted to know the total area and the length of the side CD. To consider the problem as two triangles, a line DB was drawn on the diagram.



a Find the area of triangle ABD.

b Find the distance BD.

c Find the angle BDC.

d Find the angle DBC

e Find the length DC.

f Find the area of triangle BCD.

g What is the total area of the property?

Give lengths and areas correct to 2 decimal places, and angles correct to 1 decimal place.

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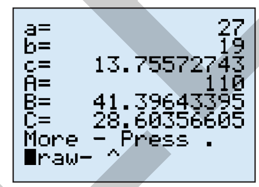
Technology tip

On the Internet you can find some excellent TI-83 Plus and TI-84 Plus programs for solving non-right-angled triangles. Make sure, however, that you test any program using a wide variety of problems, as some programs available on the Internet are faulty.

The program TRISOLVE by Ross Levine at www.ticalc.org/pub/83plus/basic/math/ completely solves any triangle when you enter the known sides or angles. Enter zero for the unknown values. The program solves the ambiguous case of the sine rule. It also finds the perimeter and the area of each triangle solved.

For example, when $a = 27,\ b = 19$ and $A = 110°$ were entered, the unknown values were found and displayed.



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